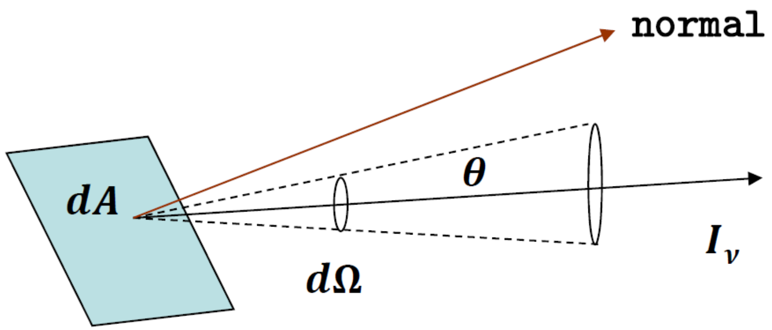
Radiative Transfer

$$I_\nu(\nu, \boldsymbol{n}, \boldsymbol{r}, t)\; d\nu\; d\Omega$$



Specific intensity (of brightness, fluence/radiant exposure)

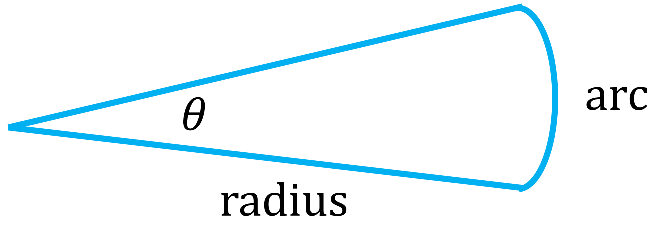
I_ν [erg s^{-1}cm^{-2}ster^{-1}Hz^{-1}] so that $\Delta E = I_\nu\, dt\, dA\, d\nu\, d\Omega$

The radiation power per unit area, with frequencies in $[\nu, \nu + d\nu]$, propagating in direction $\boldsymbol{n}$, within the solid angle $d\Omega$, including both polarizations.

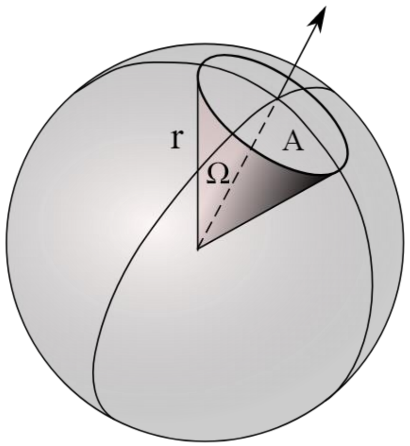
Because $\Delta\omega \rightarrow 0$, the energy does not diverge. Intensity/brightness is independent of the distance from the source (i.e., light ray).

Rybiki & Lightman

Radian: unit of a planar angle; $\theta = \text{arc}/\text{radius}$;
2π rad $= 360$ deg



Steradian (sr) : unit of a solid angle; $\Omega = \text{area}/\text{radius}$;
whole sky: 4π sr



$$1\text{ sr} = \left(\frac{180}{\pi}\right)^2 \approx 3283\text{ deg}^2$$

Entire sky $\approx$ 41,253 deg^2

Mean Intensity $J_\nu = \frac{1}{4\pi}\int I_\nu \, d\Omega$

Net Flux F_ν [ergs s^{-1} cm^{-2} Hz^{-1}]

$F_\nu = \int I_\nu \cos\theta \, d\Omega$

Total Flux $F = \int F_\nu \, d\nu$

Thermodynamic equilibrium = no net matter or energy flow into a system

Two systems in thermal equilibrium when T the same
Two systems in mechanical equilibrium when P the same
Two systems in diffusive equilibrium when μ chemical potentials the same

Momentum Flux

For photons, $dp_\nu = dF_\nu / c$

p_ν [dynes cm^{-2} Hz^{-1} $= \frac{1}{c} \int I_\nu \cos^2\theta \, d\Omega$

Momentum Flux Rate = Pressure

$P =$ [force]/[area] $= m \cdot a_\perp/\text{area} = m\frac{dv_\perp}{dt}/\text{area} = \frac{dp_\perp}{dt dA}$

Energy Density

u_ν [ergs cm^{-3} Hz^{-1}] $= \frac{1}{c} \int I_\nu \, d\Omega = \frac{4\pi}{c} J_\nu$

Total Energy Density $u = \int u_\nu \, d\nu = a \, T^4$ Stefan-Boltzmann law

$a = \frac{4\sigma}{c} = 7.56 \times 10^{-15}$ ergs cm^{-3} K^{-4}

Radiation Pressure

Each quantum of energy, $E = h\nu$, there is associated a momentum $h\nu/c$

Radiation pressure → net rate of momentum transfer

(cf. gas pressure)

Radiation passing per second through a unit area at an angle with the normal, in a solid angle $d\omega$ is $I \cos\theta \, d\omega$

→ Momentum transfer = $(I \cos\theta \, d\omega/c) \cos\theta$

projection of the area

normal to the surface

$$\therefore P_R = \frac{2}{c} \int I \cos^2\theta \, \mathrm{d}\omega$$

For isotropic radiation, $P_R = \frac{4\pi I}{3c} = u/3 = aT^4/3$

Blackbody Radiation

$$B_\nu(\mathrm{T})\, d\nu = \frac{2\,h\nu^3}{c^2}\,\frac{1}{e^{h\nu/kT}-1}\,d\nu$$ (Planck's law)

Energy density $u(\nu, T)d\nu = \frac{4\pi}{c} I = \frac{8\pi\, h}{c^3}\,\frac{\nu^3}{e^{h\nu/kT}-1}\,d\nu$

Total Energy $u = \int u(\nu, T)\, d\nu$, $u = aT^4$ (Stefan-Boltzmann law)

In terms of wavelength,

$$|d\nu| = c\frac{d\lambda}{\lambda^2}$$

$$B_\lambda(\mathrm{T})\, d\lambda = \frac{2\,hc^2}{\lambda^5}\,\frac{1}{e^{hc/\lambda kT}-1}\,d\lambda$$ (Planck's law)

When $h\nu / kT \gg 1$

$$B_\nu(\mathrm{T})\, d\nu \approx \frac{2\, h\nu^3}{c^2} e^{-h\nu/kT}\, d\nu$$ (Wien approximation)

When $h\nu / kT \ll 1$, (long wavelength or high temperature, valid in almost all radio regimes in astronomical conditions)

$$B_\nu(\mathrm{T})\, d\nu \approx \frac{2\, h\nu^3}{c^2} \frac{kT}{h\nu}\, d\nu = \frac{2kT}{c^2} \nu^2\, d\nu = \frac{2kT}{\lambda^2}\, d\nu$$

$e^x \approx 1 + x + \cdots$

(Rayleigh-Jeans approximation)

Because $B_\nu \propto T$, radio astronomy → brightness temperature … T_{antenna}, T_{noise}, etc. … even if radiation is <u>not</u> thermal.

Absorption

Consider radiation through a slab of thickness dx, the intensity is reduced by an amount

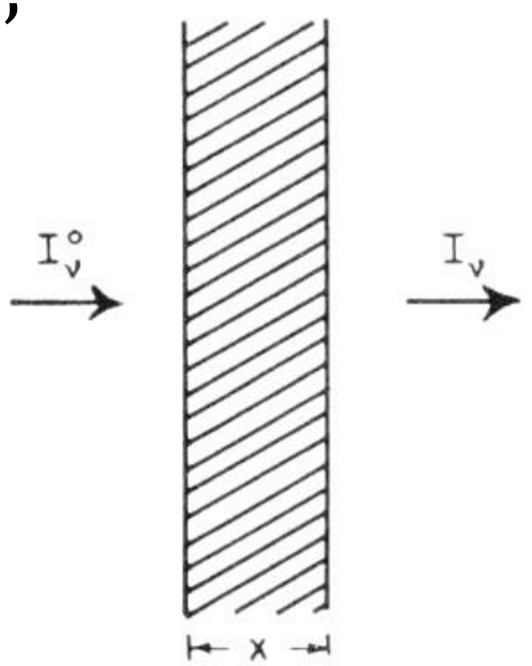
$$dI_\nu = -\kappa'_\nu \, \rho \, I_\nu \, ds \; .... \; (1)$$

Absorption coefficient κ_ν [cm^{-1}]

$$dI_\nu = -\kappa \, I_\nu \, ds$$

or κ'_ν [cm^2 g^{-1}] $\rightarrow$ **mass absorption coefficient**

This is opacity, i.e., what causes absorption lines.



I^0_ν: incident beam

Dividing (1) by I_ν and integrating

$\rightarrow$ $\ln I_\nu = -\kappa_\nu \, I_\nu \, s + \text{const}$

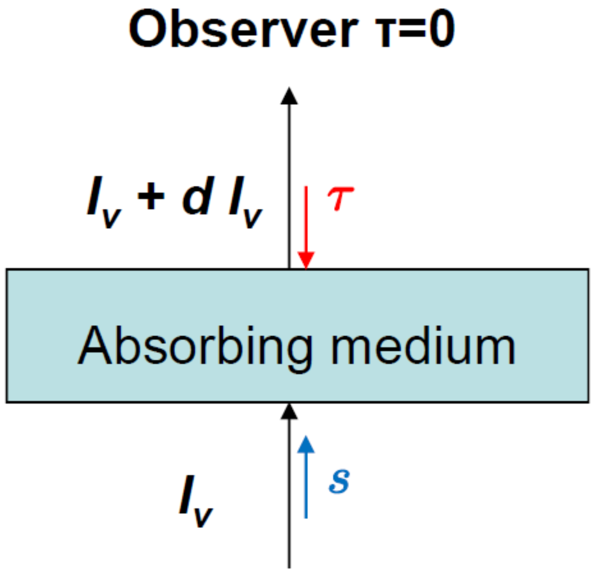
$$I_\nu = I^0_\nu \; e^{-\kappa_\nu s}$$

Introducing (dimensionless) **optical depth** τ_ν

$$d\tau_\nu = -\kappa_\nu \; ds$$

Or $\tau_\nu = \int_{s_0}^{s} \kappa_\nu(s') \; ds'$

we get $\quad I_\nu = I_\nu^0 \; e^{-\tau_\nu}$



Optical thickness:

- ✓ $\tau_\nu \gg 1$ → optically thick = opaque
- ✓ $\tau_\nu \ll 1$ → optically thin = transparent

$\tau_\nu \equiv 1$ → "surface", $1/e$ (37%) of emerging radiation

When $\kappa_\nu^{\mathrm{abs}}$ and $\kappa_\nu^{\mathrm{sca}}$ are independent of ν, the opacities are gray.

Why is the sky blue? Why is a cloudy sky gray?

Emission

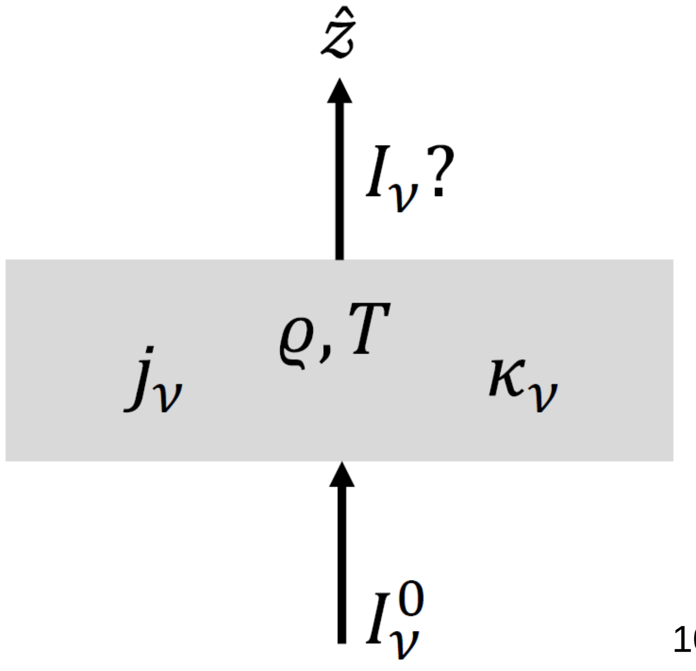
$j_\nu \, dt \, dV \, d\omega \, d\nu =$ Energy emitted

$\kappa_\nu I_\nu \, dt \, dV \, d\omega \, d\nu =$ Energy absorbed

Spontaneous emission coefficient = Emissivity

j_ν [ergs s^{-1} cm^{-3} ster^{-1} Hz^{-1}]

$dI_\nu = j_\nu \, ds$, $\hat{s}$ along the line of sight



Radiative Transfer Equation

How specific intensity varies with emission and absorption by a medium

$$\frac{dI_\nu}{ds} = -\kappa_\nu\, I_\nu + j_\nu$$

If there is scattering → radiation in and out of the solid angle → an integrodifferential equation, solution complex

$$\frac{dI_\nu}{d\tau_\nu} = -\,I_\nu + \frac{j_\nu}{\kappa_\nu} = -\,I_\nu + S_\nu$$

$$\tau_\nu(s) = \int_{s_0}^{s} \kappa_\nu(s')\, ds'$$

$S_\nu \equiv \frac{j_\nu}{\kappa_\nu}$ is the **source function.**

This equation is used more often, because S_ν is a simpler function of physical quantities, and τ_ν is more intuitive (dimensionless).

(1) $\kappa_\nu = 0$ (emission only)

$$I_\nu(s) = I_\nu(s_0) + \int_{s_0}^{s} j_\nu(s')\, ds'$$

Increase in brightness equals to the emission coefficient integrated along the line of sight.

(2) $j_\nu = 0$ (absorption only)

$$I_\nu(s) = I_\nu(s_0) \exp\left[- \int_{s_0}^{s} \kappa_\nu(s')\, ds'\right]$$

Brightness decreases exponentially by the absorption coefficient integrated along the line of sight.

$$\frac{dI_\nu}{d\tau_\nu} = -\,I_\nu + S_\nu$$

(3) In general

$$I_\nu(\tau_\nu) = I_\nu(0)\,e^{-\tau_\nu} + \int_0^{\tau_\nu} \frac{j_\nu}{\kappa_\nu} e^{\tau_\nu''} d\tau_\nu''$$

If $j_\nu/\kappa_\nu = $ const (not valid in ISM but OK in stellar atmosphere), then

$$I_\nu(\tau_\nu) = I_\nu(0)\,e^{-\tau_\nu} + \frac{j_\nu}{\kappa_\nu}(1 - e^{-\tau_\nu}\,)$$

In LTE, $dI_\nu/d\tau = 0 \rightarrow I_\nu = j_\nu/\kappa_\nu$ and $I_\nu = B_\nu(T)$

$$\frac{dI_\nu}{d\tau_\nu} = -I_\nu + S_\nu$$

$$j_\nu = B_\nu\,\kappa_\nu$$

(Kirchhoff's law) cf Kirchhoff's circuit law

Finally

$$I_\nu(\tau_\nu) = I_\nu(0)\,e^{-\tau_\nu} + B_\nu(T)\,(1 - e^{-\tau_\nu})$$

<u>Note</u>: Assumptions of (1) LTE, and (2) $T = \text{const}$

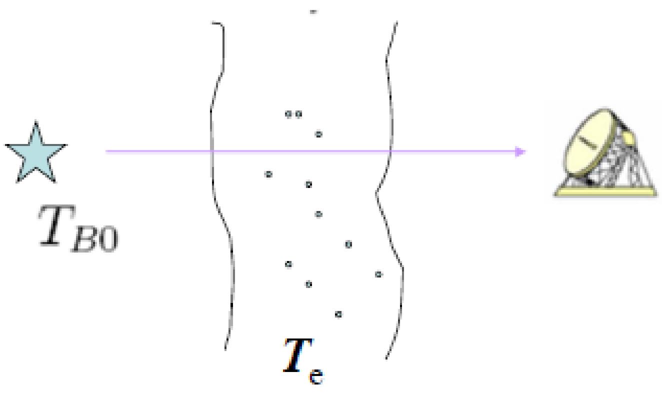
Here T is the electron temperature, T_e (ISM)
In radio, intensity → brightness temperature, T_b (signal)

$$I_\nu = (2k\nu^2/c^2)\,T_b$$

$I_\nu(\tau_\nu) = I_\nu(0)\, e^{-\tau_\nu} + B_\nu(T)\,(1 - e^{-\tau_\nu})$

In Rayleigh-Jeans regime, $B_\nu \leftrightarrow T_e$, and $I \leftrightarrow T$

$$T_B = T_B(0)\, e^{-\tau_\nu} + T_e\,(1 - e^{-\tau_\nu})$$



If background is zero ($T_B(0) = 0$), dropping ν,

(i) $\tau \gg 1 \longrightarrow T_B \rightarrow T_e$ (measures only the “surface”)

(ii) $\tau \ll 1 \longrightarrow T_B \rightarrow \tau\, T_e$ (measures the entire medium)

$I_\nu(\tau_\nu) = I_\nu(0)\, e^{-\tau_\nu} + B_\nu(T)\,(1 - e^{-\tau_\nu})$

What we actually measure is the **flux density**,

$$S_\nu \equiv \int_{\text{source}} I_\nu(\theta, \phi)\, \cos\theta \; d\Omega$$

If the source angular size is small $\ll$ 1 rad, $\cos\theta \approx 1$,

$$S_\nu = \int_{\text{source}} I_\nu \; \mathrm{d}\Omega \; [\text{ergs s}^{-1}\ \text{cm}^{-2}\ \text{Hz}^{-1}]$$

Integrating over the solid angle subtended by the source,

$$S_\nu = \int_{\text{source}}^{\Omega} B_\nu(T_e)\,(1 - e^{-\tau_\nu})\; \mathrm{d}\omega \approx \Omega\, B_\nu(T_e)\,(1 - e^{-\tau_\nu})$$

$S_\nu \propto \text{distance}^{-2}$

spectral luminosity $L_\nu = 4\pi d^2 S_\nu$

bolometric luminosity $L = \int_0^\infty L_\nu \; d\nu$

1 jansky (a spectral flux density, spectral irradiance)

$$1 \text{ Jy} = 10^{-23} \text{ ergs s}^{-1} \text{ cm}^{-2} \text{ Hz}^{-1} = 10^{-26} \text{ watts m}^{-2} \text{ Hz}^{-1}$$

Giant radio solar bursts $10^8 - 10^9$ Jy; other strong sources $\sim 10^4$ Jy; typically a few Jy; state-of-the-art a few mJy

$$\text{AB magnitude} = -2.5 \log_{10}\left(\frac{S_\nu}{3631 \text{ Jy}}\right)$$

$$S_\nu^{\text{V}=0} = 3953 \text{ Jy}$$