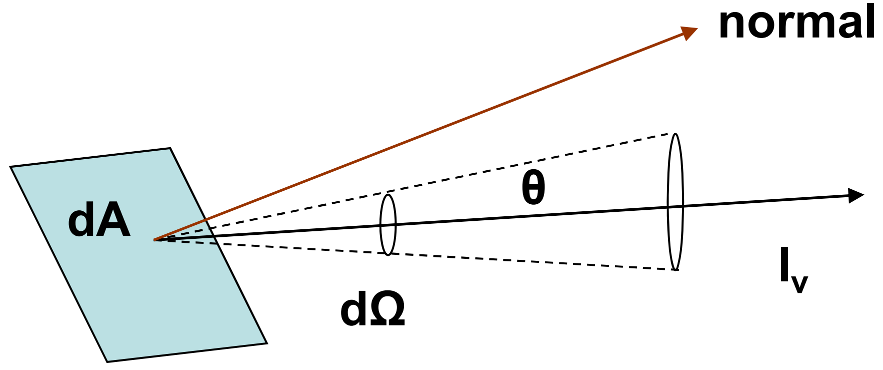
Radiative Transfer

Ref: Rybiki & Lightman



Specific Intensity (or Brightness, Fluence) I_ν

I_ν [ergs s^{-1} cm^{-2} ster^{-1} Hz^{-1}]

$$\Delta E = I_\nu \, dA \, dt \, d\Omega \, d\nu$$

$$I_\nu(\nu, \mathbf{n}, \mathbf{r}, t) \, d\nu \, d\Omega$$

The EM power per unit area, with frequencies in [ν, ν+dν] propagating in direction **n** within the solid angle dΩ, including both polarizations.

Mean Intensity $J_\nu = \frac{1}{4\pi} \int I_\nu \, d\Omega$

Net Flux F_ν [ergs s^{-1} cm^{-2} Hz^{-1}]

$F_\nu = \int I_\nu \cos\theta \, d\Omega$

Total Flux $F = \int F_\nu \, d\nu$

In local thermodynamic equilibrium (LTE)

$$(I_\nu)_{\mathrm{LTE}} \to B_\nu(T) \equiv \frac{2h\nu^3}{c^2} \frac{1}{\exp(h\nu/kT) - 1}$$

where $n\gamma(\nu) \equiv c^2/(2h\nu^3)\, I\nu$ (dimensionless) is called the **photon occupation number** = number of photons per mode of polarization

Momentum Flux

For photons, $dp_\nu = dF_\nu / c$

p_ν [dynes cm^{-2} Hz^{-1} $= \frac{1}{c} \int I_\nu \cos^2\theta \, d\Omega$

Momentum Flux Rate = Pressure

$P = [\text{force}]/[\text{area}] = m \cdot a_\perp/\text{area} = m\frac{dv_\perp}{dt}/\text{area} = \frac{dp_\perp}{dt dA}$

Energy Density

u_ν [ergs cm^{-3} Hz^{-1}] $= \frac{1}{c} \int I_\nu \, d\Omega = \frac{4\pi}{c} J_\nu$

Total Energy Density $u = \int u_\nu \, d\nu = a T^4$ **Stefan-Boltzmann law**

$a = \frac{4\sigma}{c} = 7.56 \times 10^{-15}$ ergs cm^{-3} K^{-4}

Entropy of blackbody radiation $S=(4/3)\, a\, T^3\, V$

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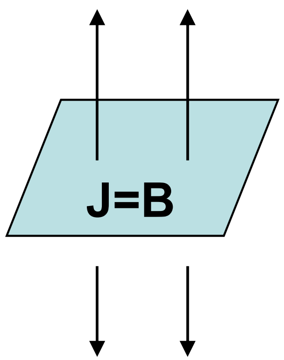
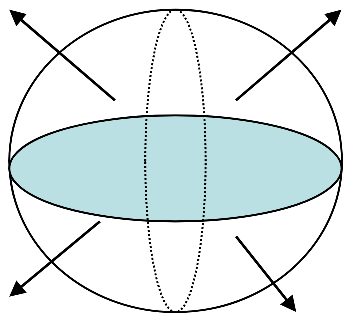
Adiabatic expansion → $PV^{4/3}$ = constant. This is the adiabatic law with $\gamma = 4/3$.

Note: $u = \frac{4\sigma}{c}\,T^4 = 4\,F/c = \frac{4\pi}{c}$

$\rightarrow F = \pi J$ for a uniform-brightness surface





$F = \pi B$

For an isolated radiation field, elastic incidence, isotropic
$I_\nu = J_\nu$

$$P_\nu = \frac{2}{c} \int I_\nu \, \cos^2\theta \, d\Omega$$

$$P = \frac{2}{c} \int J_\nu \, d\nu \, 2\pi \int_0^{2\pi} \sin\theta \, \cos^2\theta \, d\theta = \frac{2}{c}\frac{uc}{4\pi} \cdots = \frac{1}{3}\,u$$

$P = 1/3\ u$

Ohm's law $\rightarrow I = \frac{V}{R}$

i.e., $\boxed{\text{[current] = [potential difference] / [resistance]}}$

For conduction, $\frac{dQ}{dt} = -\kappa A \frac{dT}{dx}$

i.e., [heat rate] $\leftarrow$ [temp. gradient]

For radiative transfer, pressure

$$P = u/3 = a\,T^4/3$$

$$dP/dr = \frac{4}{3} a\,T^3 \frac{dT}{dr}$$

Energy flow = luminosity

$$\boxed{L \propto \frac{4}{3} a\,T^3 \frac{dT}{dr} / \bar{\kappa}\rho}$$

$\leftarrow$ opacity as resistance

Spontaneous emission coefficient = Emissivity

j_ν [ergs s^{-1} cm^{-3} ster^{-1} Hz^{-1}]

$dI_\nu = j_\nu \, ds$, $\hat{s}$ along the line of sight

Absorption coefficient κ_ν [cm^{-1}]

$dI_\nu = -\kappa \, I_\nu \, ds$

or κ'_ν [cm^2 g^{-1}] $\rightarrow$ **mass absorption coefficient**

This is opacity, i.e., what causes absorption lines.

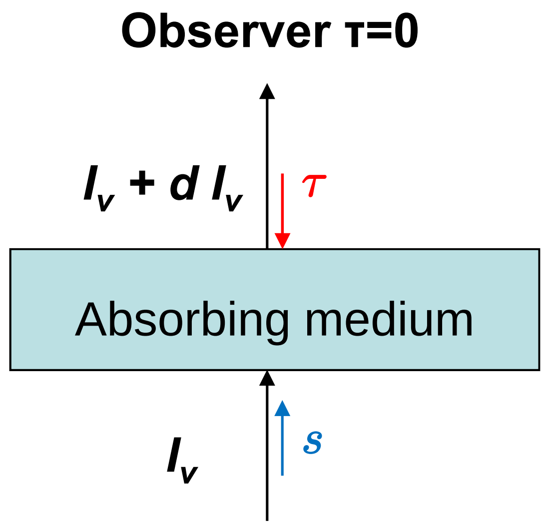
Optical Depth $d\tau_\nu = -\kappa_\nu \, ds$

or, $\tau_\nu = \int_{s_0}^{s} \kappa_\nu(s') \, ds'$

$\tau_\nu >> 1 \rightarrow$ optically think (opaque)

$\tau_\nu << 1 \rightarrow$ optically thin (transparent)

Radiative Transfer Equation



$$\frac{dI_\nu}{ds} = -\kappa_\nu I_\nu + j_\nu$$

Define $\frac{j_\nu}{\kappa_\nu} \equiv S_\nu$

$$\boxed{\frac{dI_\nu}{d\tau_\nu} = -I_\nu + S_\nu}$$

Note: The apparent surface of a star occurs at $\tau_\nu \equiv 1$, e.g., photosphere

$\tau_\nu = 1 \rightarrow 1/e(37\%)$ of radiation emerges at that point

$$\frac{dI_\nu}{d\tau_\nu} = -I_\nu + \frac{j_\nu}{\kappa_\nu}$$

$j_\nu \, dV \, d\nu \, d\Omega \, dt$: energy emitted

$\kappa_\nu \, I_\nu \, dV \, d_\nu, d\Omega \, dt$: energy absorbed

κ_ν [cm^{-1}]

(i) If $j_\nu = 0$ $\qquad I_\nu = I_\nu(0) \, e^{-\tau_\nu}$

(ii) If otherwise $\quad I_\nu(\tau) = I_\nu(0) \, e^{-\tau_\nu} + \int_0^{\tau_\nu} \frac{j_\nu}{\kappa_\nu} e^{-\tau''_\nu} \, d\tau''_\nu$

If j_ν/κ_ν = const (not valid in ISM but ok in stellar atmosphere), then

$$I_\nu(\tau) = I_\nu(0) \, e^{-\tau_\nu} + \frac{j_\nu}{\kappa_\nu} (1 - e^{-\tau_\nu})$$

In LTE, $\frac{dI_\nu}{d\tau_\nu} = 0 \rightarrow I_\nu = j_\nu / \kappa_\nu$

and $I_\nu = B_\nu(T) = \frac{2h\nu^3}{c^2} \frac{1}{e^{h\nu/kT} - 1}$

$$j_\nu = B_\nu \, \kappa_\nu$$ **Kirchhoff's law**

Finally,

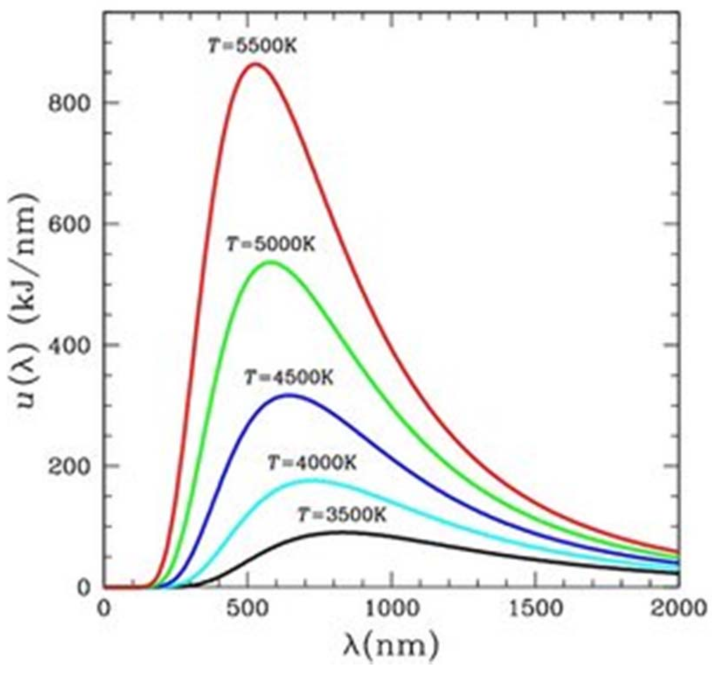
$$I_\nu(\tau) = I_\nu(0)\, e^{-\tau_\nu} + B_\nu(T)\,(1 - e^{-\tau_\nu})$$

<u>Note</u>: Assumptions (1) LTE, (2) $T = \text{const.}$

Planck Function — Approximations

$$B_\nu(T) = \frac{2h\nu^3}{c^2}\frac{1}{e^{h\nu/kT}-1} \quad [\text{ergs s}^{-1}\ \text{cm}^{-2}\ \text{ster}^{-1}\ \text{Hz}^{-1}]$$

$$B_\lambda(T) = \frac{2hc^2}{\lambda^5}\frac{1}{e^{hc/\lambda kT}-1} \quad [\text{ergs s}^{-1}\ \text{cm}^{-2}\ \text{ster}^{-1}\ \text{cm}^{-1}]$$



Wien's Displacement Law

$\lambda_{max}\, T = 2900$ [μm K]

Note: $\lambda = c/\nu, d\lambda = c d\nu/\nu^2, B_\lambda |d\lambda| = B_\nu d\nu$

$h\nu/kT \simeq 1439/\lambda_{\mu m} T_K$

Limiting Cases

(i) Rayleigh-Jeans Approximation, $h\nu/kT << 1$
i.e., long λ or high T, valid virtually in all radio regimes

$$B_\nu(T) \simeq \frac{2\nu^2}{c^2} k\,T$$

(ii) Wien's Approximation, $h\nu/kT >> 1$

$$B_\nu(T) \simeq \frac{2h\nu^3}{c^2} e^{-h\nu/kT}$$

In radio wavelengths, Rayleigh-Jeans approx.
$\rightarrow B_\nu \propto T_e$ (Brightness temperature)

Even if the radiation is NOT thermal.

Brightness Temperature $T_B(\nu) \equiv \frac{h\nu k}{\ln[1+2h\nu^3/c^2 I\nu]}$

the temperature such that a blackbody at that temperature would have specific intensity $B\nu(T_B) = I\nu$ $T_B(\nu)$ is a nonlinear function of intensity.

Antenna Temperature $T_A(\nu) \equiv \frac{c^2}{2k\nu^2} I\nu$ $T_A(\nu)$ is linear in the intensity.

In the limit $kT_A \gg h\nu$, common in radio frequencies, $T_A \approx T_B$.

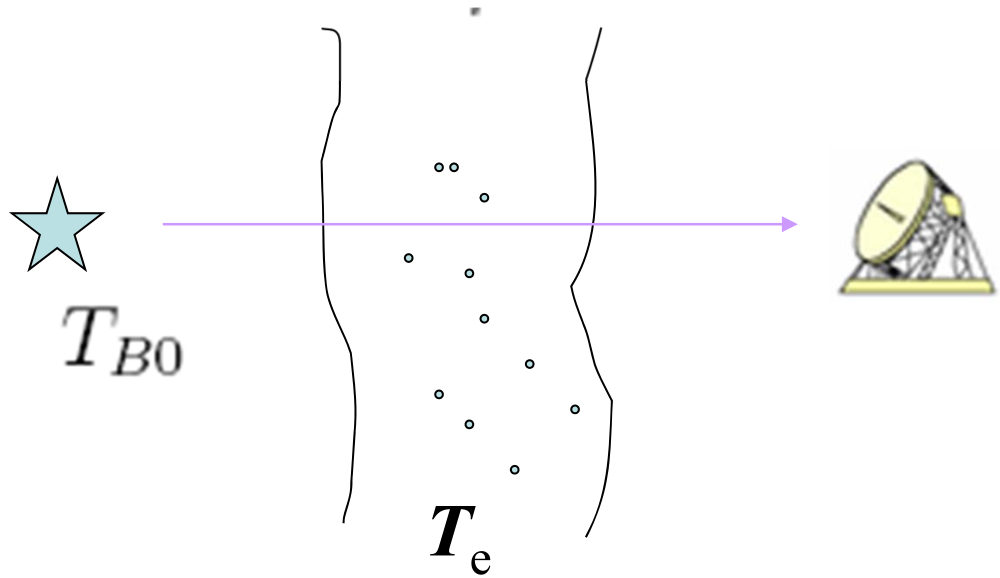
In Rayleigh-Jeans regime, $B_\nu \leftrightarrow T_e, I \leftrightarrow T$

$$T_B = T_{B0}\, e^{-\tau_\nu} + T_e\,(1 - e^{-\tau_\nu})$$

If background is zero $(T_{B0} = 0)$

(i) $\tau >> 1 \longrightarrow T_B \rightarrow T$

(ii) $\tau << 1 \longrightarrow T_B \rightarrow T\,\tau$



What we really measure is the flux density

$S_\nu = \int_{\text{source}} I_\nu \, d\Omega$ [ergs s^{-1} cm^{-2} Hz^{-1}]

integrating over the solid angle subtended by the source

$$1 \text{ Jansky} = 10^{-23} \text{ ergs s}^{-1} \text{ cm}^{-2} \text{ Hz}^{-1}$$

$$S_\nu^{\text{V=0}} = 3953 \text{ Jansky}$$

$$S_\nu = \int B_\nu(T_e)(1 - e^{-\tau_\nu}) \, d\Omega \approx \Omega \, B_\nu(T_e) \, (1 - e^{-\tau_\nu})$$