Emission and Absorption

Two ways to decay from an excited state

1. $X_2 \rightarrow X_1 + h\nu$

   **spontaneous emission**

   occurrence rate $\leftrightarrow$ atomic properties

2. $X_2 + h\nu \rightarrow X_1 + 2h\nu$

   **stimulated emission**

   occurrence rate $\leftrightarrow$ density of incoming photons of the same $\nu$, polarization, and direction of propagation

$\Delta E = h\nu$
Einstein Coefficients by Einstein (1917) regarding radiation probabilities

**Spontaneous emission**

\[ A_{21} \quad [s^{-1}] \]

\( j \quad A_{21} \quad \text{--- probability} \)

**Stimulated (induced) emission**

\[ B_{21} \quad [cm^3 \ ergs^{-1} \ s^{-1} \ Hz^{-1}] \]

\( B \ u_\nu \quad \text{--- probability} \)

**Stimulated absorption**

\[ B_{12} \]

\( k \)
**Transition Probability**

Considering a 2-level system, we want to calculate the emission arising from this transition, \( j_\nu \) [ergs s\(^{-1}\) cm\(^{-3}\) ster\(^{-1}\) Hz\(^{-1}\)].

\( j_\nu \) is governed by a distribution function (line profile)

\[
\dot{j}_\nu = \frac{h\nu A_{21} n_2 \phi_\nu}{4\pi}
\]

\[\phi_\nu\]

\[j = \int j_\nu \, d\nu, \int \phi_\nu \, d\nu = 1\]

\[j = \frac{h\nu A_{21} n_2}{4\pi}\] volume emissivity

\(A_{21}\): transition probability (per unit time) \(\approx 10^{-15}\) s\(^{-1}\) for H I 21 cm line

Assuming \(j_\nu \propto (\theta, \phi)\)
Energy absorbed in a line [ergs s\(^{-1}\) cm\(^{-3}\) ster\(^{-1}\)]

\[\int \kappa_\nu I_\nu \, d\nu \simeq I_\nu \int \kappa_\nu \, d\nu\]

This is valid for a sharp line, i.e., \(\kappa_\nu \approx \delta\) function

Emission probability: \(A\)
Absorption probability: \(B \, u_\nu = B \frac{I_\nu}{c}\)

\[
\kappa_\nu = \frac{h \nu (n_1 B_{12} - n_2 B_{21})}{c} \, \phi_\nu \quad \rightarrow \quad \int \kappa_\nu \, d\nu = \frac{h \nu (n_1 B_{12} - n_2 B_{21})}{c}
\]
In equilibrium, **detailed balance** (equal probabilities) gives

\[
\frac{h\nu n_2 A_{21}}{4\pi} = \frac{h\nu I_\nu}{c} (n_1 B_{12} - n_2 B_{21})
\]

\[
\frac{n_2 A_{21}}{4\pi} = \frac{n_1 B_{12} - n_2 B_{21}}{c} \frac{2h\nu^3}{c^2} \frac{1}{e^{h\nu/kT} - 1}
\]

\[
\frac{n_2 A_{21}}{n_1 4\pi} = \frac{B_{12} - \frac{n_2}{n_1} B_{21}}{c} \frac{2h\nu^3}{c^2} \frac{1}{e^{h\nu/kT} - 1}
\]

In case of TE, \( \frac{n_2}{n_1} = \frac{g_2}{g_1} e^{-h\nu/kT} \)

\[
\frac{g_2}{g_1} e^{-h\nu/kT} \frac{A_{21}}{4\pi} = \frac{B_{12} - \frac{g_2}{g_1} e^{-h\nu/kT} B_{21}}{c} \frac{2h\nu^3}{c^2} \frac{1}{e^{h\nu/kT} - 1}
\]

\[
= \frac{2h\nu^3}{c^3} \left[ B_{12} - \frac{g_2}{g_1} e^{-h\nu/kT} \cdot B_{21} \right] \frac{1}{e^{h\nu/kT} - 1}
\]

\[
= \frac{2h\nu^3}{c^3} e^{-h\nu/kT} \left[ B_{12} e^{h\nu/kT} - \frac{g_2}{g_1} \cdot B_{21} \right] \frac{1}{e^{h\nu/kT} - 1}
\]
If this is to hold true for any T, the 

\[ e^{\frac{\hbar \nu}{kT}} - 1 \]

term must cancel out, i.e.,

\[ g_1 B_{12} = g_2 B_{21} \]

and

\[ A_{21} = \frac{8\pi \hbar \nu^3}{c^3} B_{21} \]

In fact, the Einstein coefficients are properties of atoms, so the relations hold whether it is TE or not.
**Line Shapes**

- **Natural broadening**: uncertainty principle
- **Doppler broadening**: random thermal velocities of particles
- **Pressure broadening**: interruption of radiation train (usually not important in ISM)
- **Opacity broadening**: photons at the line wings have smaller reabsorption probabilities than those near the line center (line-of-sight effect)
Lorentzian Profile

\[ I_\nu \propto \frac{1}{(\nu - \nu_0)^2 + \left(\frac{\gamma}{4\pi}\right)^2} \]

Peaked at \( \nu_0 \) and width measured by \( \gamma \).

\[ (\Delta \nu)_{\text{intrinsic FWHM}} = \frac{\gamma 21}{2\pi} \]

Also known as the Cauchy probability distribution; the solution of the differential equation describing forced resonance.

The Lorentzian profile = an accurate approximation to the actual line profile; more accurately by the Kramers-Heisenberg formula.
Doppler Profile

Emitted light $\nu_0$ shifted to $\nu$ due to $\nu_z$

$$\frac{\nu - \nu_0}{\nu_0} = \frac{\nu_z}{c}$$

range of frequencies $\leftrightarrow$ range of velocities

For a Maxwellian distribution, $I_{\nu} \propto e^{-(\nu - \nu_0)^2/2\sigma^2}$, where

$$\sigma^2 = \frac{(\nu_0^2 k T_{\text{kin}})}{(m c^2)}$$

T↑ ➔ velocity dispersion↑

$$\left(\Delta \nu\right)_{\text{FWHM}} = \sqrt{8 \ln 2} \sigma_v$$

In general in a turbulent medium,

$$2\sigma^2 = \frac{\nu_0^2}{c^2} \left(\frac{2k T_{\text{kin}}}{m} + \nu_{\text{tub}}^2 + \cdots\right)$$
Natural Broadening

Described by the Lorentzian profile

Line very narrow; usually an order of magnitude or less than other effects, e.g., intensity drops to 2\% of peak at 0.003 Å from line center

\[ 2 \gamma \leftrightarrow A, \text{ where } 1/A = \text{time in the upper level} \]

(uncertainty principle \( \Delta E \Delta t < h \), i.e., \( h\Delta\nu (1/A) < h \))

At optical frequencies (\( \nu \sim 10^{15} \)) a typical strong line has

\[ A \sim 10^8 \text{ s}^{-1} \quad \rightarrow \Delta\nu/\nu \sim 10^{-7} \]
The nature width can be expressed in terms of the line-of-sight velocity, so as to compare with Doppler width, for example:

$$(\Delta \nu)^{\text{intrinsic}}_{\text{FWHM}} = c \frac{(\Delta \nu)^{\text{intrinsic}}_{\text{FWHM}}}{\nu_{21}} = \frac{\lambda_{21} \gamma_{21}}{2\pi}$$

the natural width for H Ly $\alpha$, $h \nu = (3/4) 13.6$ eV, $f_{12} = 0.4162$, $g_1/g_2 = 2/6$, so the intrinsic $(\Delta \nu) = 0.0121$ km s$^{-1}$
Voigt Profile

\[ V(x; \sigma, \gamma) = \int_{-\infty}^{\infty} D(x'; \sigma) L(x - x'; \gamma) \, dx' \]

\[ D(x; \sigma) = \frac{e^{-x^2/2\sigma^2}}{\sigma \sqrt{2\pi}} \quad L(x; \gamma) = \frac{\gamma}{\pi(x^2 + \gamma^2)} \]

- Convolution of a Doppler profile (= Gaussian) and a Lorentzian profile (Gaussian core + Lorentzian wings)

- The Doppler profile is more strongly peaked

- Away from the line center (i.e., \( |v - v_0| \uparrow \rightarrow \) wings), Lorentzian \( \uparrow \)

- Gaussian core: FWHM=2.3556 \( \sigma \)
Pressure (collisional) Broadening

- Profile similar to Lorentzian, with width $1/\tau_0$
  where $\tau_0$ is the mean time interval between collisions, which in the ISM is about 1000 years

→ narrower even than the natural broadening

- Pressure broadening therefore is not important in the ISM but important in stellar atmosphere where collisions are frequent.
The damping profile

\[ \kappa_\nu = \frac{h\nu (n_1B_{12} - n_2B_{21})}{c} \phi_\nu = \frac{h\nu}{c n_1B_{12}} \left(1 - \frac{n_2}{n_1} \frac{B_{21}}{B_{12}}\right) \phi_\nu = \frac{h\nu}{c n_1B_{12}} \left(1 - \frac{n_2}{n_1} \frac{g_1}{g_2}\right) \phi_\nu \]

So, if in TE \[ \kappa_\nu = \frac{h\nu n_1B_{12}}{c} \left(1 - e^{-h\nu/kT}\right) \phi_\nu \]

If non-TE, then define \( b_j \equiv \frac{n_j \text{(actual)}}{n_j \text{(in TE)}} \)

In LTE, \( b_j = 1 \)

\[ \frac{n_2}{n_1} = \frac{b_2}{b_1} \left(\frac{n_2}{n_1}\right)_{eq} = \frac{b_2}{b_1} \frac{g_2}{g_1} e^{-h\nu/kT} \]

So in general

\[ \kappa_\nu = \frac{h\nu n_1B_{12}}{c} \left(1 - \frac{b_2}{b_1} e^{-h\nu/KT}\right) \phi_\nu \]

Computation of \( b \)'s is not trivial.

If \( h\nu >> kT \) stimulating emission is negligible.
Define $\sigma_\nu$, the absorption cross section per particle

\[ \kappa_\nu \ [\text{cm}^{-1} \ \text{Hz}^{-1}] = n \ [\text{cm}^{-3}] \quad \sigma_\nu \ [\text{cm}^2 \ \text{Hz}^{-1}] \]

\[ \sigma_\nu = \sigma_0 \left[1 - \frac{b_2}{b_1} e^{\frac{h\nu}{kT}}\right] \phi_\nu \]

where $\sigma_0 = \hbar \nu B_{12}/c$

Classically $\sigma_\nu = \frac{\pi e^2}{mc} f \phi_\nu$ and $\sigma_0 = \frac{\pi e^2}{mc} f$ where $f$ is oscillator strength, is the effective number of electrons per atom.  

so,  
\[ \frac{\pi e^2}{mc} f_{12} = \frac{\hbar \nu B_{12}}{c} = \frac{\hbar \nu g_2}{c \ g_1} \frac{c^3}{8\pi\hbar \nu^3} A_{12} \]

\[ f_{12} = \frac{mc^3}{8\pi \nu^2 e^2} \frac{g_2}{g_1} A_{21} \quad A_{12} = \frac{0.6670 \ \text{cm}^2 \ \text{s}^{-1}}{\chi^2_{12}} \frac{g_1}{g_2} f_{12} \]

\[ \sigma_0 = \frac{c^2}{8\pi \nu^2} \frac{g_2}{g_1} A_{21} \quad g_1 f_{12} = -g_2 f_{21} \]
• Under some conditions, the upper level may be “pumped” (by collision or by radiative excitation of a higher level followed by a decay).
• If pumping more rapid than depopulation
  \[ n_2 / n_1 > g_2 / g_1 \] (excitation temperature, \( T_{21} < 0 \))
• This is called a **population inversion** \( \rightarrow \) stimulated emission is stronger than absorption \( \rightarrow \) radiation amplified. laser=Light Amplification by the Stimulated Emission of Radiation
• Such inversions have been observed in **microwave** transitions of H I, OH, and SiO (**maser**) (Elitzur 1992 ARAA)
• Maser sources can be very bright \( \rightarrow \) motion measured by interferometry, e.g., Galactic SFRs, or supermassive black hole in NGC 4258 (Herrnstein et al 1999)