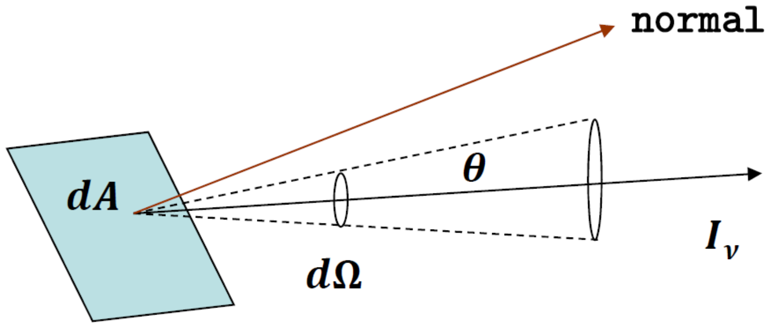
Radiative Transfer

$$I_\nu(\nu, \boldsymbol{n}, \boldsymbol{r}, t)\, d\nu\, d\Omega$$



Specific Intensity (or Brightness, Fluence)

I_ν [ergs s^{-1} cm^{-2} ster^{-1} Hz^{-1}]

$$\Delta E = I_\nu\, dA\, dt\, d\Omega\, d\nu$$

The EM power per unit area, with frequencies in $[\nu, \nu + d\nu]$ propagating in direction $\boldsymbol{n}$ within the solid angle $d\Omega$, including both polarizations.

Because $\Delta\omega \to 0$, the energy does not diverge. Intensity/brightness is independent of the distance from the source (i.e., light ray).

Rybiki & Lightman

Mean Intensity $J_\nu = \frac{1}{4\pi} \int I_\nu \, d\Omega$

Net Flux F_ν [ergs s^{-1} cm^{-2} Hz^{-1}]

$$F_\nu = \int I_\nu \cos\theta \, d\Omega$$

Total Flux $F = \int F_\nu \, d\nu$

Thermodynamic equilibrium = no net matter or energy flow into a system

Two systems in thermal equilibrium when T the same
Two systems in mechanical equilibrium when P the same
Two systems in diffusive equilibrium when μ chemical potentials the same

Momentum Flux

For photons, $dp_\nu = dF_\nu / c$

p_ν [dynes cm^{-2} Hz^{-1} $= \frac{1}{c}\int I_\nu \cos^2\theta \, d\Omega$

Momentum Flux Rate = Pressure

$P = [\text{force}]/[\text{area}] = m \cdot a_\perp/\text{area} = m\frac{dv_\perp}{dt}/\text{area} = \frac{dp_\perp}{dt dA}$

Energy Density

u_ν [ergs cm^{-3} Hz^{-1}] $= \frac{1}{c}\int I_\nu \, d\Omega = \frac{4\pi}{c} J_\nu$

Total Energy Density $u = \int u_\nu \, d\nu = a\, T^4$ Stefan-Boltzmann law

$a = \frac{4\sigma}{c} = 7.56 \times 10^{-15}$ ergs cm^{-3} K^{-4}

Radiation Pressure

Each quantum of energy, $E = h\nu$, there is associated a momentum $h\nu/c$

Radiation pressure → net rate of momentum transfer (cf. gas pressure)

Radiation passing per second through a unit area at an angle with the normal, in a solid angle $d\omega$ is $I\cos\theta\, d\omega$

→ Momentum transfer $= (I\cos\theta\, d\omega/c)\cos\theta$

projection of the area

normal to the surface

$$\therefore P_R = \frac{2}{c}\int I\cos^2\theta\, \mathrm{d}\omega$$

For isotropic radiation, $P_R = \frac{4\pi I}{3c} = u/3 = aT^4/3$

Blackbody Radiation

$$B_\nu(\mathrm{T})\,d\nu = \frac{2\,h\nu^3}{c^2}\,\frac{1}{e^{h\nu/kT}-1}\,d\nu$$ (Planck's law)

<u>Energy density</u> $u(\nu,T)d\nu = \frac{4\pi}{c}I = \frac{8\pi\,h}{c^3}\,\frac{\nu^3}{e^{h\nu/kT}-1}\,d\nu$

<u>Total Energy</u> $u = \int u(\nu,T)\,d\nu$, $u = aT^4$ (Stefan-Boltzmann law)

In terms of wavelength,

$$|d\nu| = c\frac{d\lambda}{\lambda^2}$$

$$B_\lambda(\mathrm{T})\,d\lambda = \frac{2\,hc^2}{\lambda^5}\,\frac{1}{e^{hc/\lambda kT}-1}\,d\lambda$$ (Planck's law)

When $h\nu/kT \gg 1$

$$B_\nu(\mathrm{T})\, d\nu \approx \frac{2\, h\nu^3}{c^2} e^{-h\nu/kT}\, d\nu$$ (Wien approximation)

When $h\nu/kT \ll 1$, (long wavelength or high temperature, valid in almost all radio regimes in astronomical conditions)

$$B_\nu(\mathrm{T})\, d\nu \approx \frac{2\, h\nu^3}{c^2}\frac{kT}{h\nu}\, d\nu = \frac{2kT}{c^2}\nu^2\, d\nu = \frac{2kT}{\lambda^2}\, d\nu$$

$e^x \approx 1 + x + \cdots$

(Rayleigh-Jeans approximation)

Because $B_\nu \propto T$, radio astronomy → brightness temperature … T_{antenna}, T_{noise}, etc. … even if radiation is not thermal.

Absorption

Consider radiation through a slab of thickness dx, the intensity is reduced by an amount

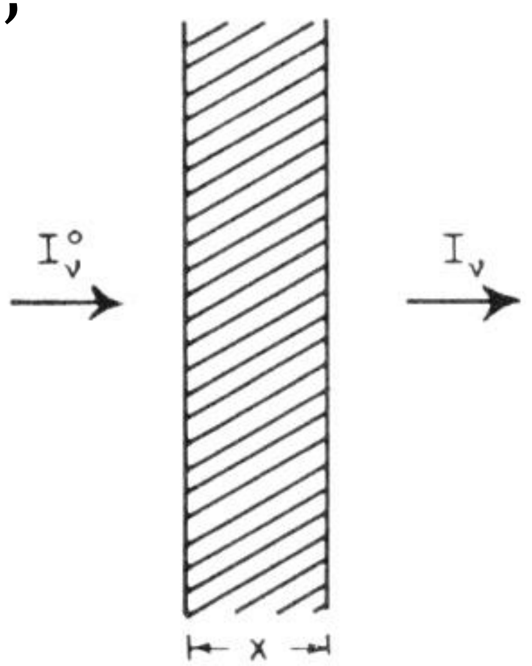
$$dI_\nu = -\kappa'_\nu\,\rho\,I_\nu\,ds \quad \ldots\ldots\ldots\ldots\ (1)$$

Absorption coefficient κ_ν [cm^{-1}]

$$dI_\nu = -\kappa\,I_\nu\,ds$$

or κ'_ν [cm^2 g^{-1}] $\rightarrow$ mass absorption coefficient

This is opacity, i.e., what causes absorption lines.



I^0_ν: incident beam

Dividing (1) by I_ν and integrating

$$\rightarrow \ln I_\nu = -\kappa_\nu\,I_\nu\,s + \text{const}$$

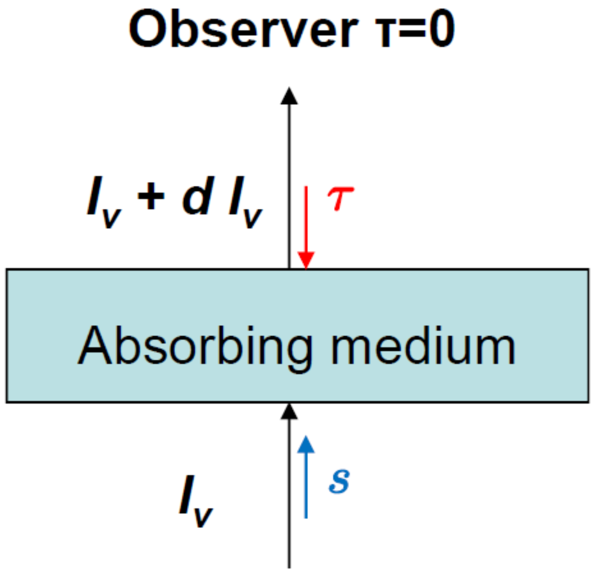
$$I_\nu = I^0_\nu\ e^{-\kappa_\nu s}$$

Introducing (dimensionless) **optical depth** τ_ν

$$d\tau_\nu = -\kappa_\nu \; ds$$

Or $\tau_\nu = \int_{s_0}^{s} \kappa_\nu(s') \; ds'$

we get $\quad I_\nu = I_\nu^0 \; e^{-\tau_\nu}$



Optical thickness:

✓ $\tau_\nu \gg 1$ → optically thick = opaque

✓ $\tau_\nu \ll 1$ → optically thin = transparent

$\tau_\nu \equiv 1$ → "surface", $1/e$ (37%) of emerging radiation

When $\kappa_\nu^{\mathrm{abs}}$ and $\kappa_\nu^{\mathrm{sca}}$ are independent of ν, the opacities are gray.

Why is the sky blue? Why is a cloudy sky gray?

Emission

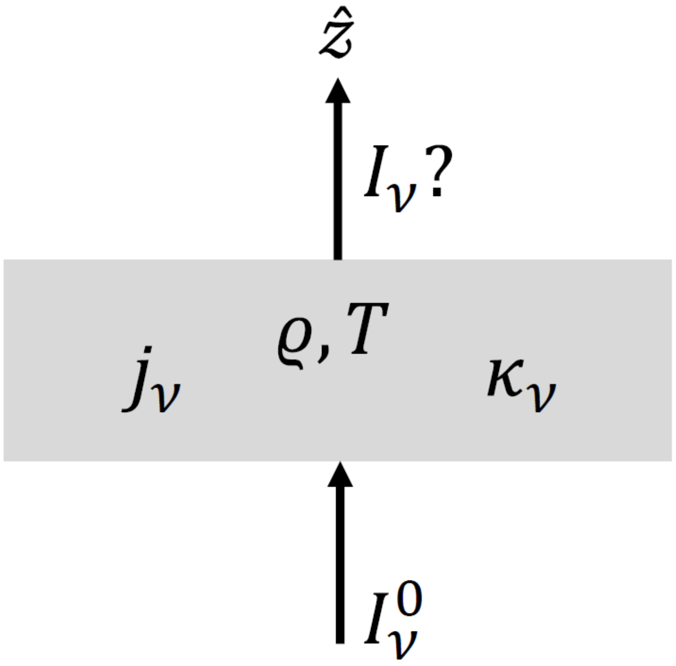
$$j_\nu \, dt \, dV \, d\omega \, d\nu = \text{Energy emitted}$$

$$\kappa_\nu I_\nu \, dt \, dV \, d\omega \, d\nu = \text{Energy absorbed}$$

Spontaneous emission coefficient = Emissivity

j_ν [ergs s^{-1} cm^{-3} ster^{-1} Hz^{-1}]

$dI_\nu = j_\nu \, ds$, $\hat{s}$ along the line of sight



Radiative Transfer Equation

How specific intensity varies with emission and absorption by a medium

$$\frac{dI_\nu}{ds} = -\kappa_\nu\, I_\nu + j_\nu$$

If there is scattering → radiation in and out of the solid angle → an integrodifferential equation, solution complex

$$\frac{dI_\nu}{d\tau_\nu} = -\,I_\nu + \frac{j_\nu}{\kappa_\nu} = -\,I_\nu + S_\nu$$

$$\tau_\nu(s) = \int_{s_0}^{s} \kappa_\nu(s')\, ds'$$

$S_\nu \equiv \frac{j_\nu}{\kappa_\nu}$ is the **source function.**

This equation is used more often, because S_ν is a simpler function of physical quantities, and τ_ν is more intuitive (dimensionless).

(1) $\kappa_\nu = 0$ (emission only)

$$I_\nu(s) = I_\nu(s_0) + \int_{s_0}^{s} j_\nu(s')\, ds'$$

Increase in brightness equals to the emission coefficient integrated along the line of sight.

(2) $j_\nu = 0$ (absorption only)

$$I_\nu(s) = I_\nu(s_0) \exp\left[-\int_{s_0}^{s} \kappa_\nu(s')\, ds'\right]$$

Brightness decreases exponentially by the absorption coefficient integrated along the line of sight.

$$\frac{dI_\nu}{d\tau_\nu} = -I_\nu + S_\nu$$

(3) In general

$$I_\nu(\tau_\nu) = I_\nu(0)\, e^{-\tau_\nu} + \int_0^{\tau_\nu} \frac{j_\nu}{\kappa_\nu} e^{\tau_\nu''} d\tau_\nu''$$

If $j_\nu/_{\kappa_\nu}$ = const (not valid in ISM but OK in stellar atmosphere), then

$$I_\nu(\tau_\nu) = I_\nu(0)\, e^{-\tau_\nu} + \frac{j_\nu}{\kappa_\nu}(1 - e^{-\tau_\nu})$$

In LTE, $dI_\nu/d\tau = 0 \rightarrow I_\nu = j_\nu/\kappa_\nu$ and $I_\nu = B_\nu(T)$

$$\frac{dI_\nu}{d\tau_\nu} = -I_\nu + S_\nu$$

$j_\nu = B_\nu\ \kappa_\nu$ (Kirchhoff's law) cf Kirchhoff's circuit law

Finally

$$I_\nu(\tau_\nu) = I_\nu(0)\ e^{-\tau_\nu} + B_\nu(T)\ (1 - e^{-\tau_\nu}\ )$$

Note: Assumptions of (1) LTE, and (2) $T = \text{const}$

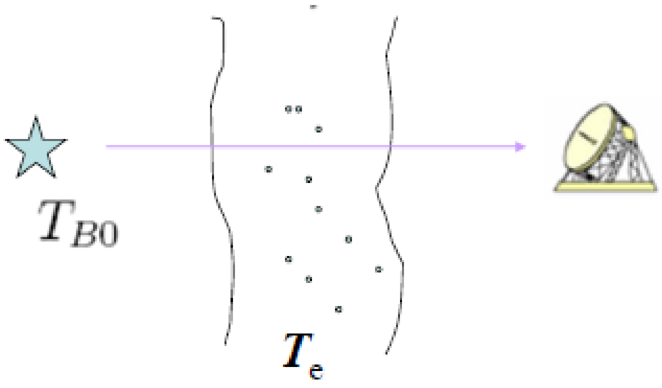
Here T is the electron temperature, T_e (ISM)
In radio, intensity → brightness temperature, T_b (signal)

$$I_\nu = (2k\nu^2/c^2)\ T_b$$

$I_\nu(\tau_\nu) = I_\nu(0)\, e^{-\tau_\nu} + B_\nu(T)\,(1 - e^{-\tau_\nu})$

In Rayleigh-Jeans regime, $B_\nu \leftrightarrow T_e$, and $I \leftrightarrow T$

$$T_B = T_B(0)\; e^{-\tau_\nu} + T_e\,(1 - e^{-\tau_\nu})$$



If background is zero ($T_B(0) = 0$), dropping ν,

(i) $\tau \gg 1 \longrightarrow T_B \rightarrow T_e$ (measures only the "surface")

(ii) $\tau \ll 1 \longrightarrow T_B \rightarrow \tau\, T_e$ (measures the entire medium)

$$I_\nu(\tau_\nu) = I_\nu(0)\, e^{-\tau_\nu} + B_\nu(T)\,(1 - e^{-\tau_\nu})$$

What we actually measure is the **flux density**,

$$\mathrm{S}_\nu \equiv \int_{\mathrm{source}} I_\nu(\theta,\phi)\ \cos\theta\ d\Omega$$

If the source angular size is small $\ll$ 1 rad, $\cos\theta \approx 1$,

$$\mathrm{S}_\nu = \int_{\mathrm{source}} I_\nu\ \mathrm{d}\Omega\ [\mathrm{ergs\ s^{-1}\ cm^{-2}\ Hz^{-1}}]$$

Integrating over the solid angle subtended by the source,

$$\mathrm{S}_\nu = \int_{\mathrm{source}}^{\Omega} B_\nu(T_e)\,(1 - e^{-\tau_\nu})\ \mathrm{d}\omega \approx \Omega\, B_\nu(T_e)\,(1 - e^{-\tau_\nu})$$

$S_\nu \propto \text{distance}^{-2}$

spectral luminosity $L_\nu = 4\pi d^2 S_\nu$

bolometric luminosity $L = \int_0^\infty L_\nu \, d\nu$

1 jansky (a spectral flux density, spectral irradiance)

$$1\ \text{Jy} = 10^{-23}\ \text{ergs s}^{-1}\ \text{cm}^{-2}\ \text{Hz}^{-1} = 10^{-26}\ \text{watts m}^{-2}\ \text{Hz}^{-1}$$

Giant radio solar bursts $10^8 - 10^9$ Jy; other strong sources $\sim 10^4$ Jy; typically a few Jy; state-of-the-art a few mJy

AB magnitude$= -2.5 \log_{10}\left(\frac{S_\nu}{3631\ \text{Jy}}\right)$

$S_\nu^{\mathrm{V}=0} = 3953\ \text{Jy}$

Exercise

Receiver noise: thermal motion of electron within the electronic circuitry

Thermal noise power per unit frequency from a resistor at temperature T_N is $p_N = k\,T_N$.

Show that at room temperature (300 K), $p_N \sim 4 \times 10^5$ Jy.

Smith

Exercise

Mayer et al (1958) measured an antenna temperature of 0.24 K at a wavelength of 3.15 cm for Mars, which subtended an angle of 18”. The radio telescope had a beam of 0.116 deg (between two half-power points). Find the equivalent temperature of Mars.

Solution. The radius of the disk of Mars is 9 sec of arc or 9/3,600 = 0.0025°. Hence, the solid angle of the disk is given by

$$\Omega_s = \pi r^2 = \pi (0.0025°)^2 = 2 \times 10^{-5} \text{ deg}^2$$

The beam area Ω_A of the antenna is given approximately by (see Chap. 6)

$$\Omega_A = \tfrac{4}{3}(0.116)^2 = 0.018 \text{ deg}^2$$

Hence, assuming a constant temperature over the disk, the average equivalent temperature of Mars by this measurement is, from (3-118),

$$T = T_A \frac{\Omega_A}{\Omega_s} = 0.24 \frac{0.018}{2 \times 10^{-5}} = 216 \text{ K}$$

Radian: unit of a planar angle; $\theta = \text{arc/radius}$;
2π rad =360 deg

Steradian (sr) : unit of a solid angle; Ω= area/radius;
4π sr=whole sky

$$1 \text{ sr} = \left(\frac{180}{\pi}\right)^2 \approx 3283 \text{ deg}^2$$

$$\text{Entire sky} \approx 41{,}253 \text{ deg}^2$$