Compact Objects

Compact Objects

Nuclear energy $\quad 4\,m_H - m_{He} = 0.029\, m_H$

$$\text{mass deficit} = 7 \times 10^{-3}\ \text{g/g}$$

$$\therefore \text{Energy available} = mc^2 = \underline{6 \times 10^{18}\ \text{erg g}^{-1}}$$

Chemical energy $\quad \lesssim 100\ \text{Kcal} \Rightarrow \underline{4 \times 10^{12}\ \text{erg g}^{-1}}$

Gravitational energy $\quad$ e.g. for $\odot$, $\dfrac{3}{5}\dfrac{M_\odot^2 G}{R_\odot} \sim 2 \times 10^{48}\ \text{erg}$

$$\Rightarrow \underline{10^{15}\ \text{erg g}^{-1}}$$

Accretion $\quad \dfrac{MG}{r}\cdot \dot{m}$

In general $\dfrac{E_{nuc}}{mass} \sim 0.01\,c^2$ $\dfrac{E_{grav}}{mass} \sim \dfrac{3GM}{5R}$,

$\uparrow\uparrow$ as $R \downarrow\downarrow$

For very compact objects, large amounts of gravitational energy can be released, perhaps even more than nuclear energy,

$$R \lesssim \frac{MG}{0.01\,c^2} \sim 10^7\,\text{cm} \sim 100\,\text{km}, \text{ for } 1\,M_\odot$$

cf. Schwarzschild radius $R_S \equiv \dfrac{2GM}{c^2} \sim 3\,\text{km}$, for $1\,M_\odot$

More about Degeneracy

Atoms in a white dwarf are fully ionized and the e^- gas is degenerate.

1844 Bessel observed the oscillated path of Sirius

1862 Sirius B discoved by Clark

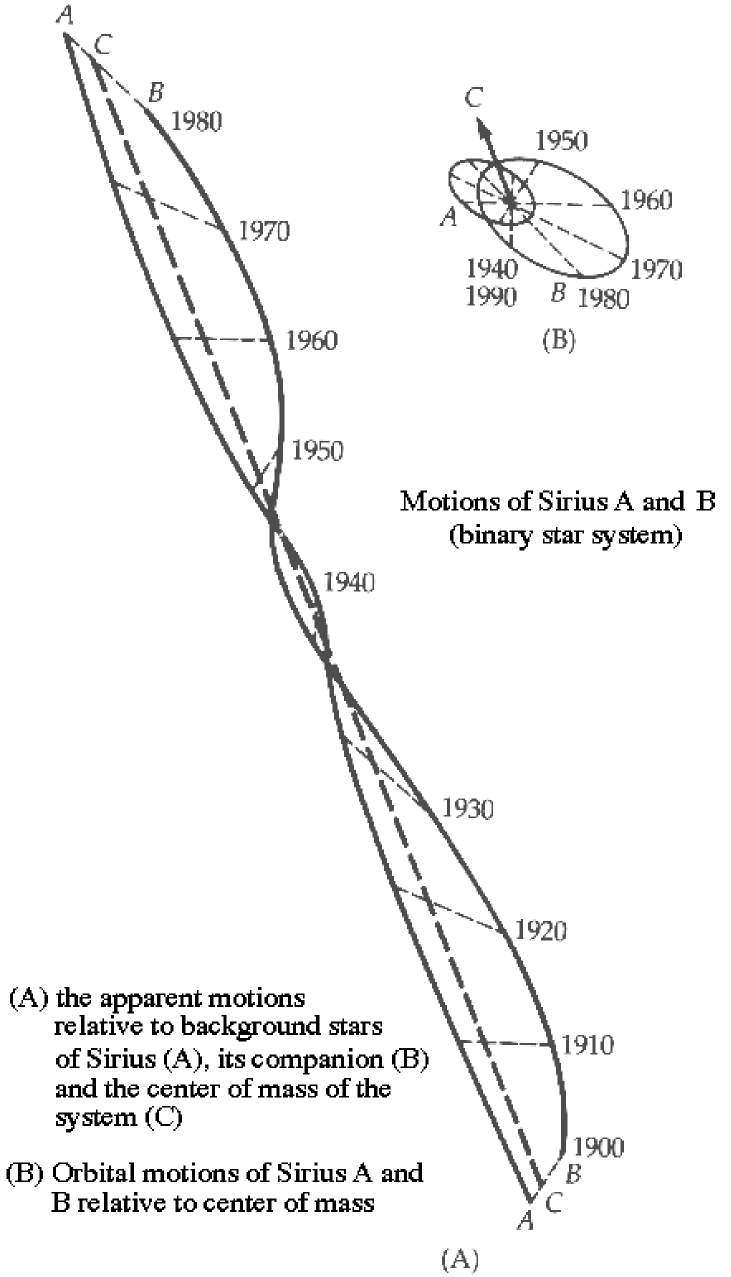
$M(\text{Sirius B}) \sim 2 \times 10^{33}\,g$ ← orbit

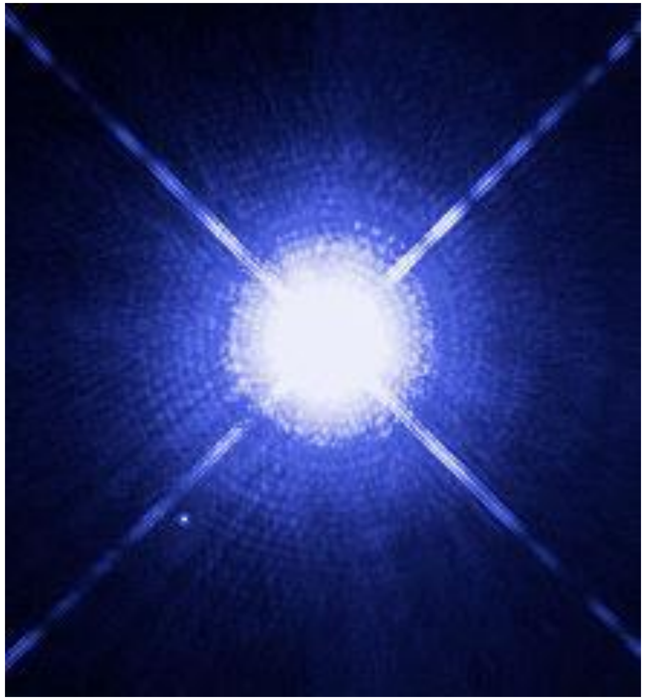
$R(\text{Sirius B}) \sim 2 \times 10^{9}\,cm$ ← surface temp. and radiation

cf $R_\odot \sim 7 \times 10^{10}\,cm$

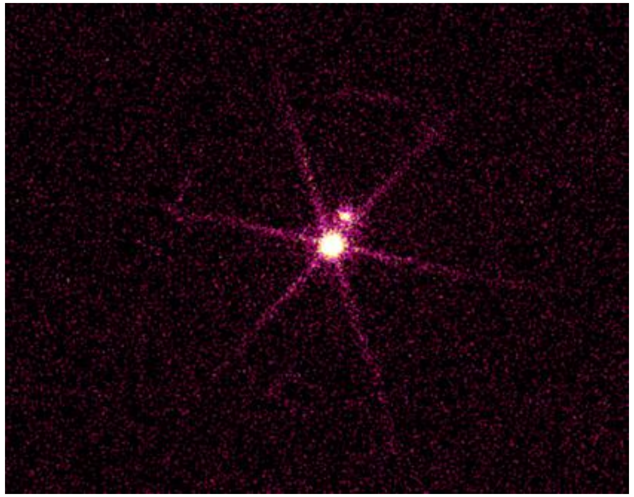
$$\bar{\rho}_{\text{Sirius B}} = \frac{M}{\frac{4}{3}\pi R^3} \sim 0.7 \times 10^{5}\,g\,cm^{-3}$$

cf $\bar{\rho}_{sun} \sim 1\,g\,cm^{-3}$





Sirius A and B by the *HST*



Sirius B and A by the *Chandra Observatory*

For WDs $\langle \rho \rangle \sim 10^5 - 10^6 \; g\,cm^{-3}$

mean separation of carbon ions

$$\langle d_{ii} \rangle \sim \left(\frac{\rho}{m_c}\right)^{-1/3} \approx 0.02\,Å$$

$$m_c \simeq 12\, m_H$$

but the size of a normal carbon atom

$$r_c \simeq \frac{a_0}{Z} \simeq \frac{a_0}{6} \simeq 0.08\,Å$$

$\therefore$ complete ionization

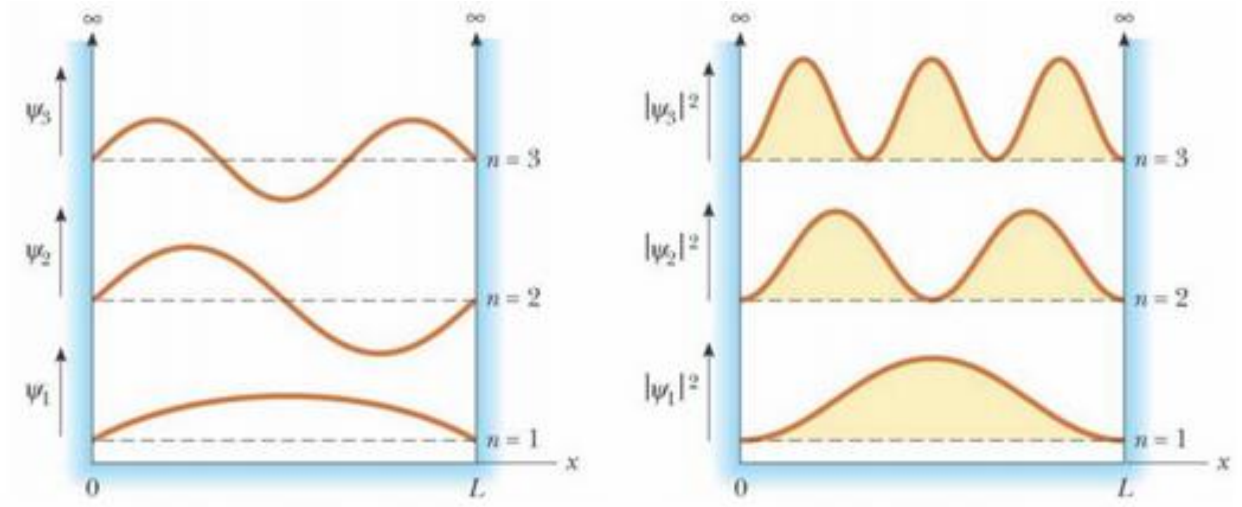
$\rightarrow$ fermion gas of separate nuclei & e^- electron gas

Mean separation of electrons

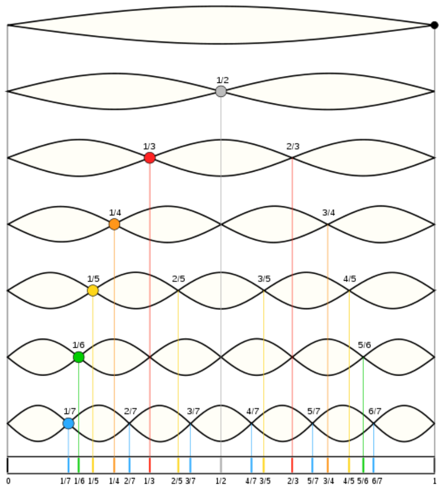
$$\langle d_{ee} \rangle \sim \left(\frac{Z\rho}{m_c}\right)^{-1/3} \approx 0.01\,Å$$

but $\lambda_e = \left[\hbar^2/m_e k T\right]^{1/2} \approx 10\,Å \Rightarrow$ QM treatment!

Particle in a Box



cf. standing wave in a string



$\Psi = 0$ at the walls

→ De Broglie wavelength

$$\lambda_n = 2L/n\,, \quad n = 1, 2, 3, \ \ldots$$

Since $\lambda_n = {}^{h}/_{mv} \rightarrow E_K = {}^{1}/_{2}\ mv^2 = (mv)^2/2m = \dfrac{h^2}{2m\lambda^2}$

No potential ➔ $E_n = (mv)^2/2m = \dfrac{h^2}{2m\lambda_n^2} = \dfrac{n^2h^2}{8mL^2} = \dfrac{1}{2m}\dfrac{n^2\pi^2\hbar^2}{L^2}$

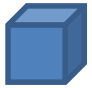
Within the box, the Schrödinger equation

$$\frac{d^2\psi}{dx^2} + \frac{2m}{\hbar^2} E\psi = 0 \rightarrow \psi_n = \sqrt{\frac{2}{L}} \sin\frac{n\pi x}{L}$$

At the <u>center</u>, ψ_1, ψ_3 probability → max
ψ_2 probability $= 0$

c.f. classical physics → same probability everywhere in the box

Consider an atom in a box of volume $V = \ell^3$



wave equation $-\dfrac{\hbar^2}{2m}\nabla^2\psi = \varepsilon\,\psi$

energies, $\varepsilon_n = \dfrac{\hbar^2}{2m}\left(\dfrac{\pi}{\ell}\right)^2\left[n_x^2 + n_y^2 + n_z^2\right]$

where n_i's are quantum no's

any positive integer

(n_i)

In the phase space

$$\varepsilon_F = \frac{\hbar^2}{2m}\left(\frac{\pi\, n_F}{\ell}\right)^2$$

n_F: radius that separates

filled & empty states

$$\varepsilon_F = \frac{\hbar^2}{2m}\left(\frac{\pi\, n_F}{\ell}\right)^2$$

For N electrons

one octant

$$N_e = 2 \times \frac{1}{8} \times \frac{4}{3}\pi n_F^3 \qquad n_F = \left(\frac{3}{\pi} N_e\right)^{1/3}$$

2 spin states

$$\therefore \quad \varepsilon_F = \frac{\hbar^2}{2m}\frac{\pi^2}{V^{2/3}}\left(\frac{3}{\pi}N_e\right)^{2/3} = \frac{\hbar^2}{2m}\left(3\pi^2\frac{N_e}{V}\right)^{2/3}$$

$$\boxed{\varepsilon_F = \frac{\hbar^2}{2m}\left(3\pi^2 n_e\right)^{2/3}} \sim n_e^{2/3}$$

electron concentration

Fermi energy: the <u>highest</u> filled energy level at temperature zero

The total energy of the system in the ground state

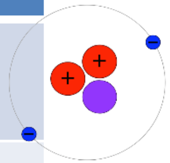
$$U_e = 2\sum_{n \le n_F} \mathcal{E}_n = 2 \times \frac{1}{8} \times 4\pi \int_0^{n_F} n^2 \mathcal{E}_n \, dn$$

$$= \frac{\pi^2}{2m}\left(\frac{\hbar}{l}\right)^2 \int_0^{n_F} n^4 \, dn \qquad \mathcal{E}_n = \frac{\hbar^2}{2m}\left(\frac{\pi n}{l}\right)^2$$

$$= \frac{\pi^3}{10m}\left(\frac{\hbar}{l}\right)^2 n_F^5 = \cdots = \frac{3}{5} N_e \mathcal{E}_F$$

Fermi energy of degenerate fermion gases

Phase of matter	Particles	E_F	$T_F = E_F/k_B$[K]
Liquid ^{3}He	atoms	4×10^{-4}eV	4.9
Metal	electrons	2–10 eV	5×10^4
White dwarfs	electrons	0.3 MeV	3×10^9
Nuclear matter	nucleons	30 MeV	3×10^{11}
Neutron stars	neutrons	300 MeV	3×10^{12}



$$\boxed{E_F = \frac{\hbar^2}{2m_e}\left(3\pi^2 n_e\right)^{2/3}}$$

For any nonrelativistic particles

$$PV = \frac{2}{3} N E_K \Rightarrow P = \frac{2}{3} n E_K$$

For nonrelativistic degenerate gas

$$E_K = \frac{3}{5} \varepsilon_F = \frac{3}{5} (3\pi^2)^{2/3} \frac{\hbar^2}{2m} n_e^{2/3}$$

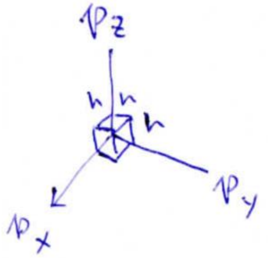
$$\Rightarrow P_{deg} \sim 1.004 \times 10^{13} \left(\frac{\rho}{\mu_e}\right)^{5/3} [\text{dynes cm}^{-2}]$$

$\mu_e \approx 2$ with no H

Degenerate State

$$E_n = \frac{\hbar^2}{2m}\left(\frac{n\pi}{l}\right)^2 \Rightarrow E_f = \frac{\hbar^2}{2m}\left(\frac{n_F\pi}{l}\right)^2 = \frac{\hbar^2}{2m}\left(3\pi^2 n_e\right)^{2/3}$$

Total $N_e = 2\cdot\frac{1}{8}\cdot\frac{4}{3}\pi n_F^3 = \frac{\pi}{3} n_F^3 \Rightarrow n_F = \left(\frac{3}{\pi} n_e\right)^{1/3}$



Uncertainty Principle $\Delta V \, \Delta^3 p \lesssim h^3$

$$2\cdot 4\pi p^2\, dp = h^3 \cdot n_e(p)\, dp$$

Considering the problem in terms of momentum

Up to p_F, $2\cdot\frac{4}{3}\pi p_F^3 = N_e = n_e\cdot h^3 \Rightarrow p_F = \left(\frac{3h^3}{8\pi} n_e\right)^{1/3}$

Pressure Integral $P = \frac{1}{3}\int_0^\infty n(p)\, v\, p\, dp$ (use $v = p/m_e$)

Non-relativistic

$$= \frac{1}{3}\int_0^{p_F} \frac{8\pi p^2}{h^3}\,\frac{p}{m_e}\, p\, dp$$

$$= \frac{8\pi}{3 m_e h^3}\,\frac{1}{5}\, p_F^5 = \frac{8\pi}{15 m_e h^3}\, p_F^5$$

For electrons, $n_e = \frac{\rho}{\mu_e m_H}$ $\quad\therefore P = \frac{h^2}{20 m_e}\left(\frac{3}{\pi}\right)^{2/3}\left(\frac{\rho}{\mu_e m_H}\right)^{5/3}$

Pressure and Momentum

$$\boldsymbol{P} = \frac{1}{3}\int_0^\infty n(p)\, v\, p\, dp$$

In the non-relativistic case

$$
\begin{aligned}
P^{\mathrm{NR}}_{\mathrm{e,deg}} &= \frac{h^2}{20 m_e}\left(\frac{3}{\pi}\right)^{2/3} \frac{1}{m_{\mathrm{H}}^{5/3}} \left(\frac{\rho}{\mu_e}\right)^{5/3} \\
&= 1.00 \times 10^{13} \left(\frac{\rho}{\mu_e}\right)^{5/3} \quad [\mathrm{cgs}] \\
&\propto \rho^{5/3}
\end{aligned}
$$

In the extremely relativistic case $v \rightarrow c$ in the pressure integral

$$
\begin{aligned}
P^{\mathrm{ER}}_{\mathrm{e,deg}} &= \frac{hc}{8}\left(\frac{3}{\pi}\right)^{1/3} \frac{1}{m_{\mathrm{H}}^{3/4}} \left(\frac{\rho}{\mu_e}\right)^{4/3} \\
&= 1.24 \times 10^{15} \left(\frac{\rho}{\mu_e}\right)^{4/3} \quad [\mathrm{cgs}] \\
&\propto \rho^{4/3}
\end{aligned}
$$

For a composition devoid of hydrogen, and not very rich in extremely heavy elements, $\mu_e \approx 2$.

Mechanical Pressure

$$P = P_{Ions} + P_{electrons} + P_{rad} + \dots$$

- If the gas nondegenerate

$$P_I + P_e = P_{gas} = \frac{k}{\mu m_H} \rho T$$

- If gas e^- degenerate $\quad P_I$: ideal gas

P_e : degenerate eq. of state

- If photon gas $\quad P_I + P_e \ll P_{rad} = \frac{1}{3} a T^4$, $\quad a = \frac{4\sigma}{c}$

$\boxed{\text{Note}}$

Above needs modifications

- $T \uparrow\uparrow$, e.g. $T > 10^9$ K

  $p^+ \cdot e^-$ pair production

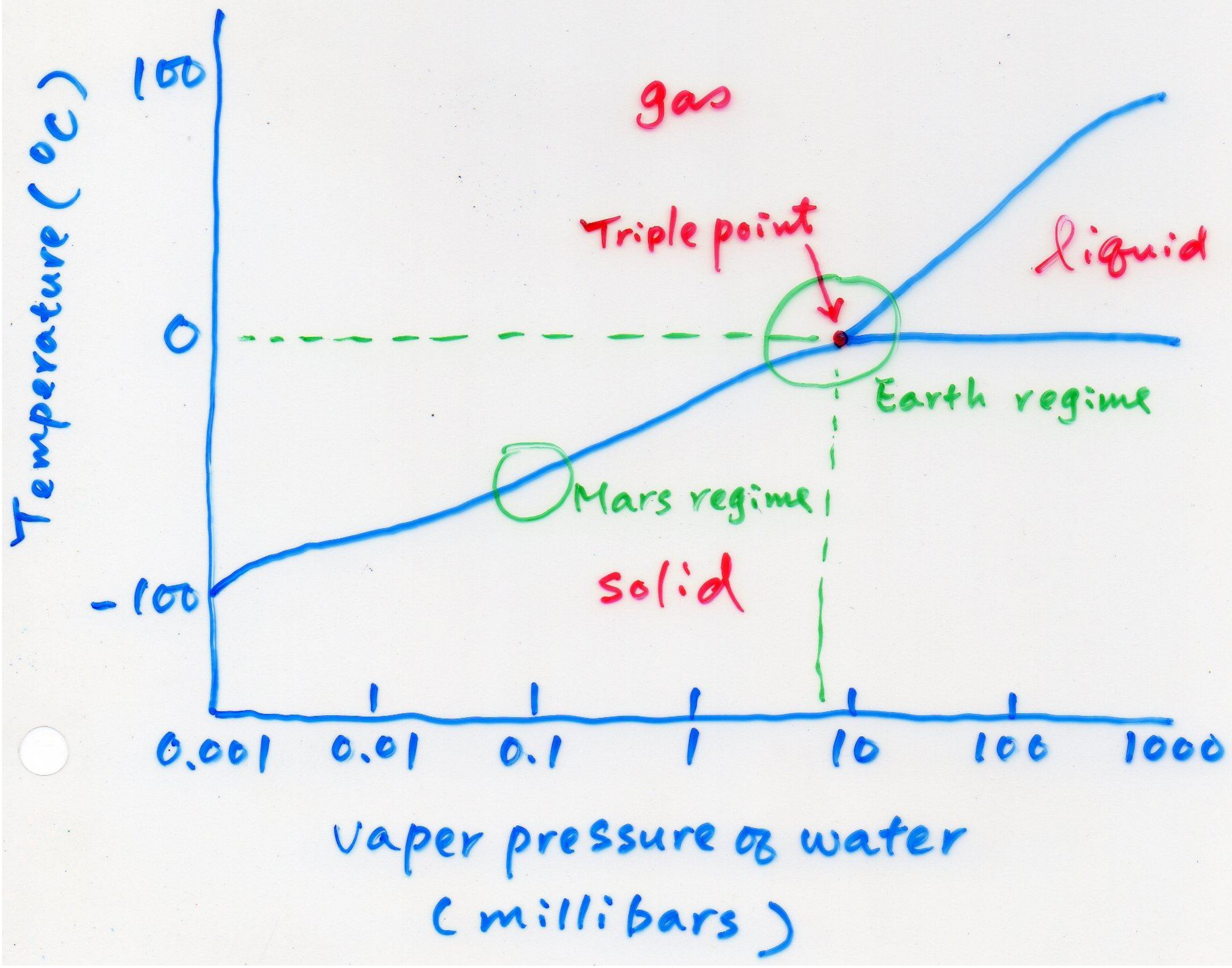
- $\rho \uparrow\uparrow$, particle interaction $\not\leftrightarrow$ ideal gas

- $\vec{B}$, addition of P_{mag}

Radiation pressure $P_{rad} = \frac{1}{3} a T^4$

For $P_{gas} = P_{rad} \Rightarrow T = 3.20 \times 10^7 (\rho/\mu)^{1/3} \sim 3.6 \times 10^7 \rho^{1/3}$

$$\boldsymbol{P}_{\text{ideal gas}} \propto \rho\, T/\mu$$


gas
Triple point
liquid
Earth regime
Mars regime
solid
Temperature (°C)
100
0
-100
0.001
0.01
0.1
1
10
100
1000
vaper pressure of water
(millibars)

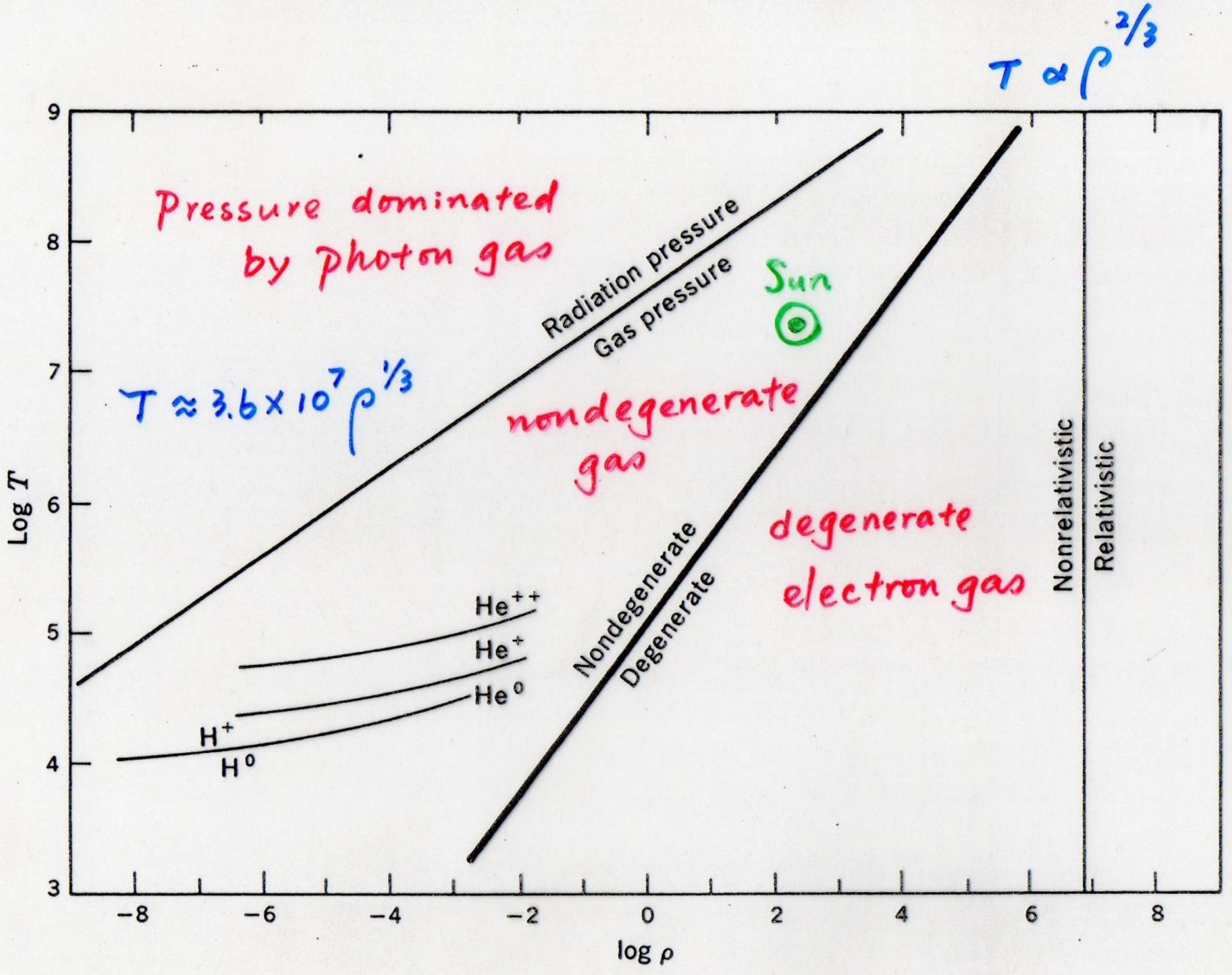


Fig. 2-11 Zones of the equation of state of a gas in thermodynamic equilibrium. Radiation pressure dominates the gas pressure in the upper left-hand corner. The remaining boundaries are similar to those in Fig. 2-7. Also included for comparison are the transition strips in a hydrogen-dominated gas between H^0 and H^+, between He^0 and He^+, and between He^+ and He^{++}.

From clayton

Nonrelativistic, complete degeneracy

- $P_{NR,e} \sim 1.004 \times 10^{13} (\rho/\mu_e)^{5/3}$ [dynes cm^{-2}]

cf NR, non-degenerate case, i.e., ideal gas

- $P_{ideal} \sim \rho T$

So, as $\rho \uparrow \Rightarrow P_{ideal} \longrightarrow P_{deg}$

and at relatively
low temperature

$$P_{gas} = P_{ions} + P_{e^-} = \left(\frac{1}{\mu_I} + \frac{1}{\mu_e}\right)\ldots$$

$$\equiv \frac{1}{\mu}\ldots$$

$$\therefore \quad \frac{1}{\mu} \equiv \frac{1}{\mu_I} + \frac{1}{\mu_e} \quad = 0.61 \text{ for } \odot$$

$$\text{cf.} \quad \frac{1}{\mu_e} \approx \frac{1}{2}(1+X) \text{ for } \odot$$

$\sum_i X_i \frac{Z_i}{A_i}$ [average # of free electrons per nucleon]

$$\left(\frac{\rho}{\mu_e}\right)^{5/3} \longleftrightarrow \rho T \quad \text{or} \quad T \sim \rho^{2/3}$$

$$\boxed{\frac{\rho_{crit}}{\mu_e} \gtrsim 2.4 \times 10^{-8}\, T^{3/2}\ [\mathrm{g\ cm^{-3}}]}$$ when degeneracy sets in

Relativistic complete degeneracy

Total energy $\sim m_0 c^2$

$p_0 c$

$$\boxed{\frac{\rho_{crit}}{\mu_e} \gtrsim 7.3 \times 10^6 \, [\mathrm{g\,cm^{-3}}]}$$

where relativistic kinetics has to be used.

Note $\rho > 10^6 \ \mathrm{g\,cm^{-3}}$ for a degerate gas to be relativistic, $T > 10^9 K$ to be completely degenerate.

Condition that satisfy both $\rho > 10^6$, $T > 10^9$ probably exist only in very late stages of stellar evolution

Almost in all other cases, nonrelativistic is ok!

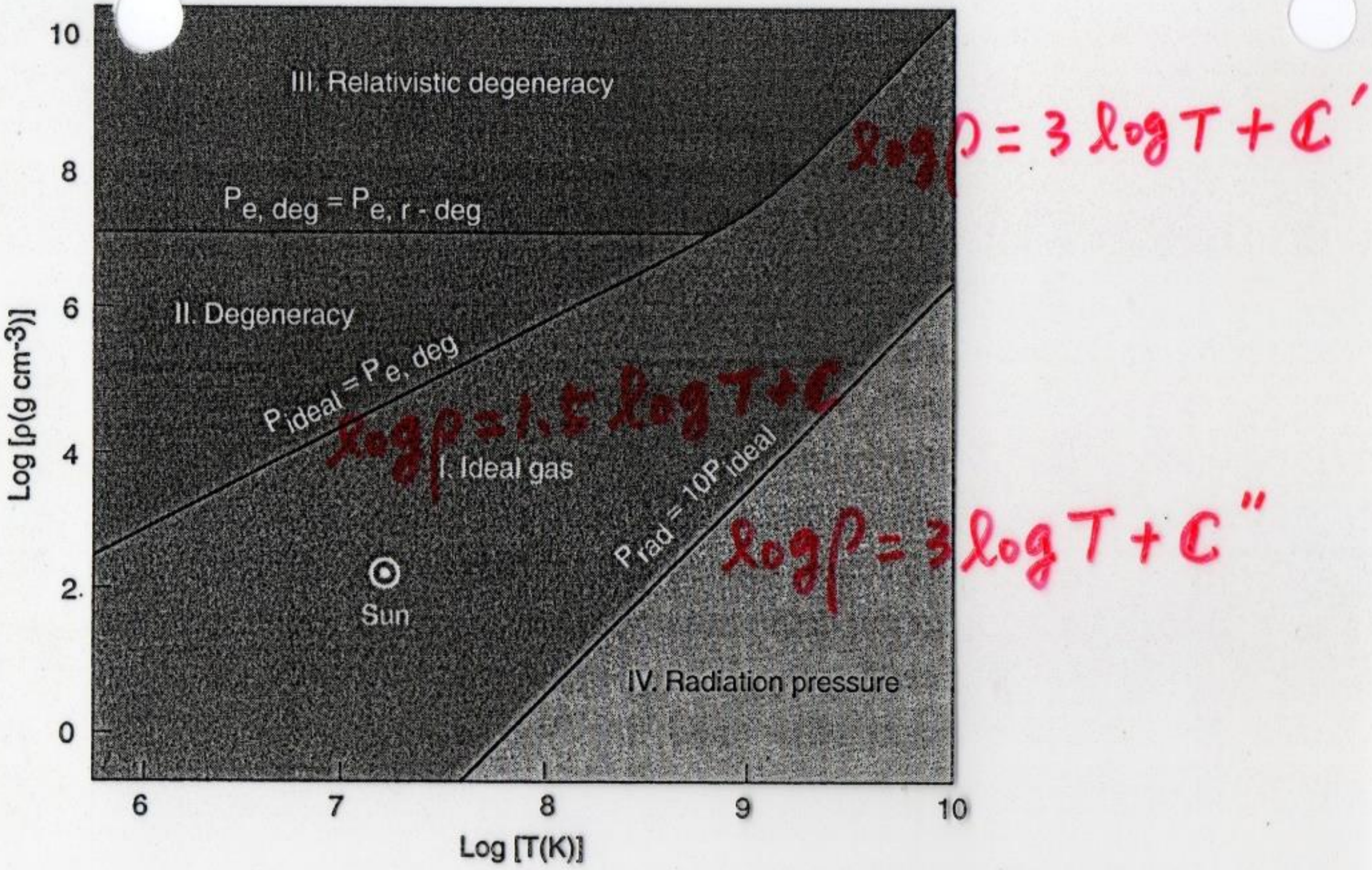


Figure 7.1 Mapping of the temperature-density diagram according to the equation of state.

In general → partial degeneracy

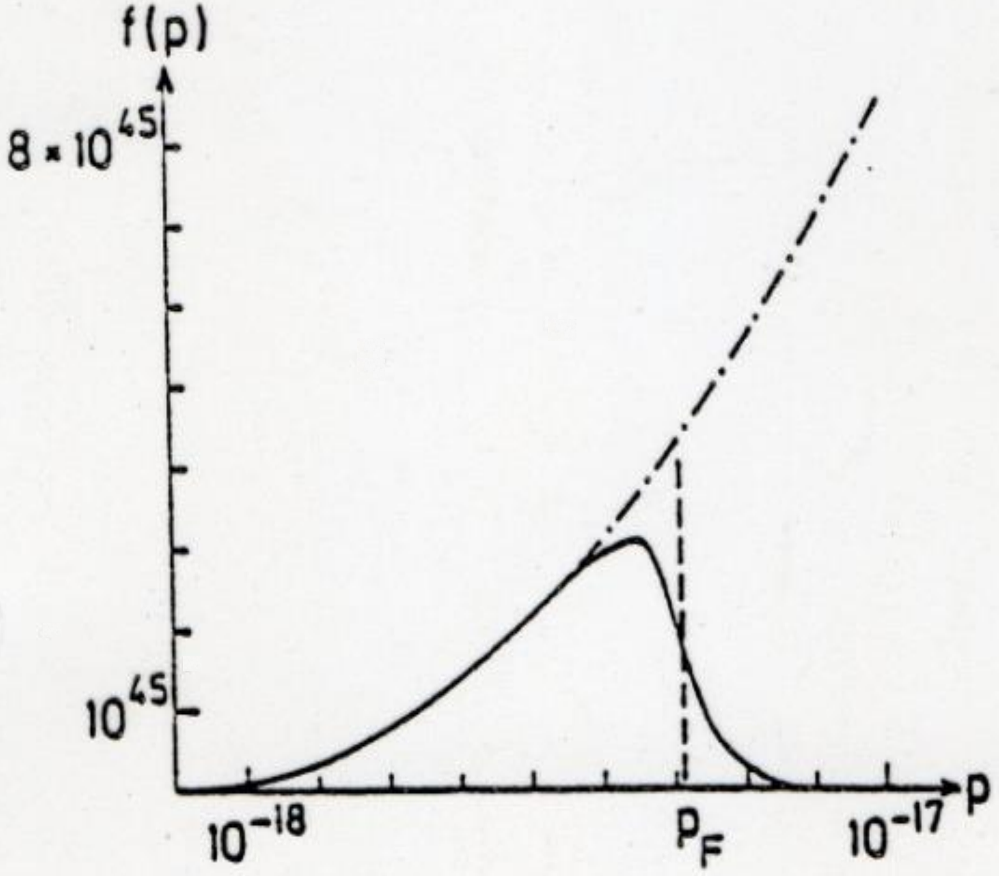


Fig. 15.5. The solid line gives the distribution function ($f(p)$ and p in cgs) for a partially degenerate electron gas with $n_e = 10^{28}$ cm^{-3} and $T = 1.9 \times 10^7$ K, which corresponds to a degeneracy parameter $\psi = 10$ (cf. the case of complete degeneracy of Fig. 15.2). The dot-dashed line shows the further increase of the parabola that defines an upper bound for the distribution function

… need evaluation of each parameter …

$$n_e = \frac{8\pi}{h^3}\int_0^\infty \frac{p^2\,dp}{1+\exp\left[E/RT - \psi\right]}$$

$$P_e = \frac{8\pi}{3h^3}\int_0^\infty p^3 \cdot v(p)\,\frac{dp}{1+\exp\left[\frac{E}{RT} - \psi\right]}$$

$$u_e = \frac{8\pi}{h^3}\int_0^\infty \frac{E\,p^2\,dp}{1+\exp\left[\frac{E}{RT} - \psi\right]}$$

In the non-rel. case $E = P^2/2m_e$

$$n_e = \frac{8\pi}{h^3}\int \frac{P^2\,dP}{1+\exp\left[\frac{P^2}{2m_e kT} - \psi\right]} \equiv \frac{8\pi}{h^3}(2m_e kT)^{3/2} a(\psi)$$

where $a(\psi) = \displaystyle\int_0^\infty \frac{\eta^2}{1+\exp[\eta^2 - \psi]}\,d\eta$

where $\eta \equiv P/(2m_e kT)^{1/2}$

Note : $n_e \sim T^{3/2} a(\psi)$

so, $\psi \equiv \psi(n_e T^{-3/2})$

$\vdots$

(rel. case 略)

Define Fermi-Dirac Integral

$$F_\nu(\psi) = \int_0^\infty \frac{u^\nu}{1+e^{u-\psi}}\,du$$

$$n_e = \frac{4\pi}{h^3}(2m_e kT)^{3/2} F_{1/2}(\psi)$$

In general, the condition may be neither highly relativistic, nor completely nonrelativistic.

The pressure can be expressed as

$$P = K f(x)$$

$$f(x) = \cdots \qquad x = p_F/m_e c$$

Tabulation of Fermi integrals

Table 15.1 Numerical values for Fermi–Dirac functions $F_{1/2}$, $F_{3/2}$ (after McDOUGALL, STONER, 1939) F_2, F_3 (after HILLEBRANDT, 1989)

Ψ	$\frac{2}{3}F_{3/2}(\Psi)$	$F_{1/2}(\Psi)$	$F_2(\Psi)$	$F_3(\Psi)$
−4.0	0.016179	0.016128	0.036551	0.109798
−3.5	0.026620	0.026480	0.060174	0.180893
−3.0	0.043741	0.043366	0.098972	0.297881
−2.5	0.071720	0.070724	0.162540	0.490154
−2.0	0.117200	0.114588	0.266290	0.805534
−1.5	0.190515	0.183802	0.434606	1.321232
−1.0	0.307232	0.290501	0.705194	2.160415
−0.5	0.489773	0.449793	1.134471	3.516135
0.0	0.768536	0.678094	1.803249	5.683710
0.5	1.181862	0.990209	2.821225	9.100943
1.0	1.774455	1.396375	4.328723	14.393188
1.5	2.594650	1.900833	6.494957	22.418411
2.0	3.691502	2.502458	9.513530	34.307416
2.5	5.112536	3.196598	13.596760	51.496218
3.0	6.902476	3.976985	18.970286	75.749976
3.5	9.102801	4.837066	25.868717	109.179565
4.0	11.751801	5.770726	34.532481	154.252522
4.5	14.88489	6.77257	45.20569	213.80007
5.0	18.53496	7.83797	58.13474	291.02151
5.5	22.73279	8.96299	73.56744	389.48695
6.0	27.50733	10.14428	91.75247	513.13900
6.5	32.88598	11.37898	112.93904	666.29376
7.0	38.89481	12.66464	137.37668	853.64147
7.5	45.55875	13.99910	165.31509	1080.24689
8.0	52.90173	15.38048	197.00413	1351.54950
8.5	60.94678	16.80714	232.69369	1673.36371
9.0	69.71616	18.27756	272.63375	2051.87884
9.5	79.23141	19.79041	317.07428	2493.65928
10.0	89.51344	21.34447	366.26528	3005.64445
10.5	100.58256	22.93862	420.45675	3595.14883
11.0	112.45857	24.57184	479.89871	4269.86200
11.5	125.16076	26.24319	544.84118	5037.84863
12.0	138.70797	27.95178	615.53418	5907.54847
12.5	153.11861	29.69679	692.22772	6887.77637
13.0	168.41071	31.47746	775.17183	7987.72229
13.5	184.60190	33.29308	864.61653	9216.95127
14.0	201.70950	35.14297	960.81184	10585.40346
14.5	219.75048	37.02649	1064.00779	12103.39411
15.0	238.74150	38.94304	1174.45439	13781.61356
15.5	258.69893	40.89206	1292.40167	15631.12726
16.0	279.63888	42.87300	1418.09966	17663.37576
16.5	301.57717	44.88535	1551.79837	19890.17470
17.0	324.52939	46.92862	1693.74783	22323.71482
17.5	348.51087	49.00235	1844.19805	24976.56198
18.0	373.53674	51.10608	2003.39907	27861.65710
18.5	399.62188	53.23939	2171.60091	30992.31625
19.0	426.78099	55.40187	2349.05358	34382.23057
19.5	455.02855	57.59313	2536.00711	38045.46629
20.0	484.37885	59.81279	2732.71153	41996.46477

$$P_{\text{ideal gas}} \propto \rho\, T/\mu$$

$$P_{e,deg}^{NR} = 1.00 \times 10^{13} \left(\frac{\rho}{\mu_e}\right)^{5/3} \ [\text{cgs}]$$

$$P_{e,deg}^{ER} = 1.24 \times 10^{15} \left(\frac{\rho}{\mu_e}\right)^{4/3} \ [\text{cgs}]$$

$$P_{\text{rad}} = \frac{1}{3}\, a\, T^4$$

Non-Relativistic, Non-Degenerate (i.e., ideal gas)

Non-Relativistic, Extremely Degenerate

Extremely Relativistic, Extremely Degenerate

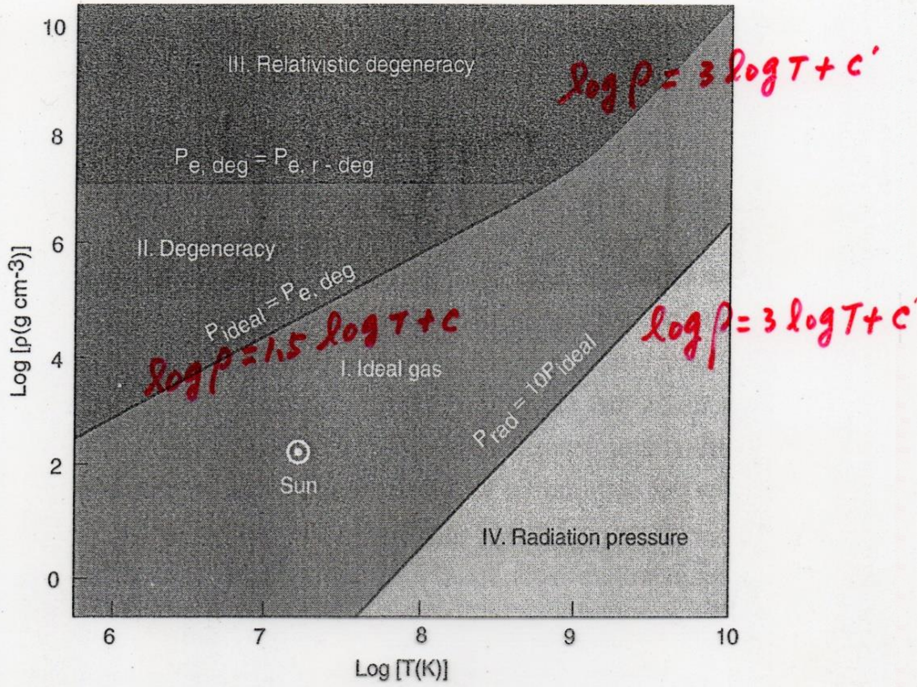


Figure 7.1 Mapping of the temperature-density diagram according to the equation of state.

NR, ND $\quad P \sim \rho T$

NR, ED $\quad P \sim \rho^{5/3}$ $\quad \Big\}\ \log\rho = 1.5 \log T + \text{const.}$

ER, ED $\quad P \sim \rho^{4/3}$ $(\sim \rho T)$ $\quad \Big\}\ \log\rho = 3 \log T + \text{const}$

P_{rad} vs $P_{ideal\ gas}$ $\quad P_{rad} \sim T^4$, $P_{gas} \sim \rho T$ $\quad \Big\}\ \log\rho = 3 \log T + \text{const}$

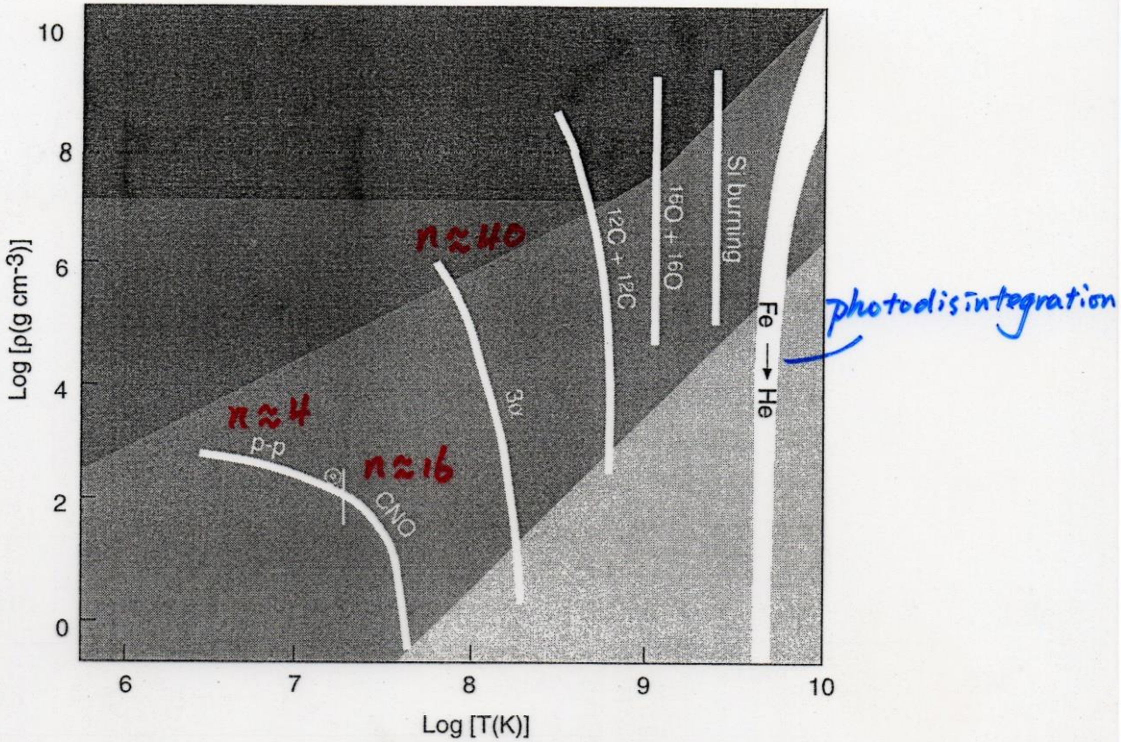


Figure 7.2 Mapping of the temperature-density diagram according to nuclear processes.

$q = q_0 \rho^m T^n$ threshold e.g.,

$q > q_{min} \left(\equiv 10^3 \ \mathrm{erg\, s^{-1} g^{-1}}\right)$

$\Rightarrow$ important

$$\log \frac{q_{min}}{q_0} = m \log \rho + n \log T$$

$$\Rightarrow \quad \log \rho = \underbrace{-\frac{n}{m}}_{\text{slope} < 0} \log T + \frac{1}{m} \log \left(q_{min} / q_0 \right)$$

For H (p-p, CNO), He (3α), C, O, Si burning, $n \gg m$

$\Rightarrow$ nearly vertical lines

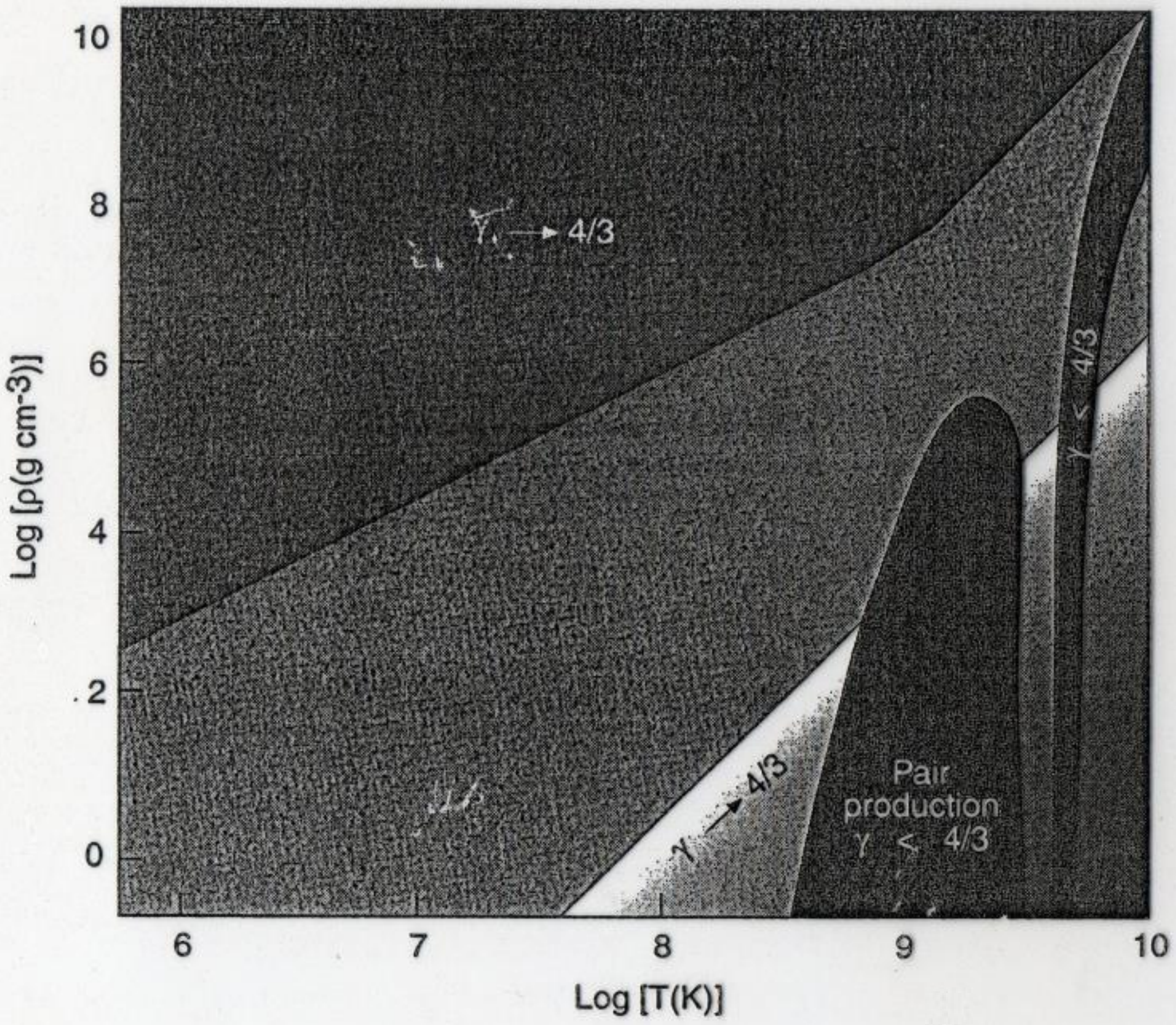
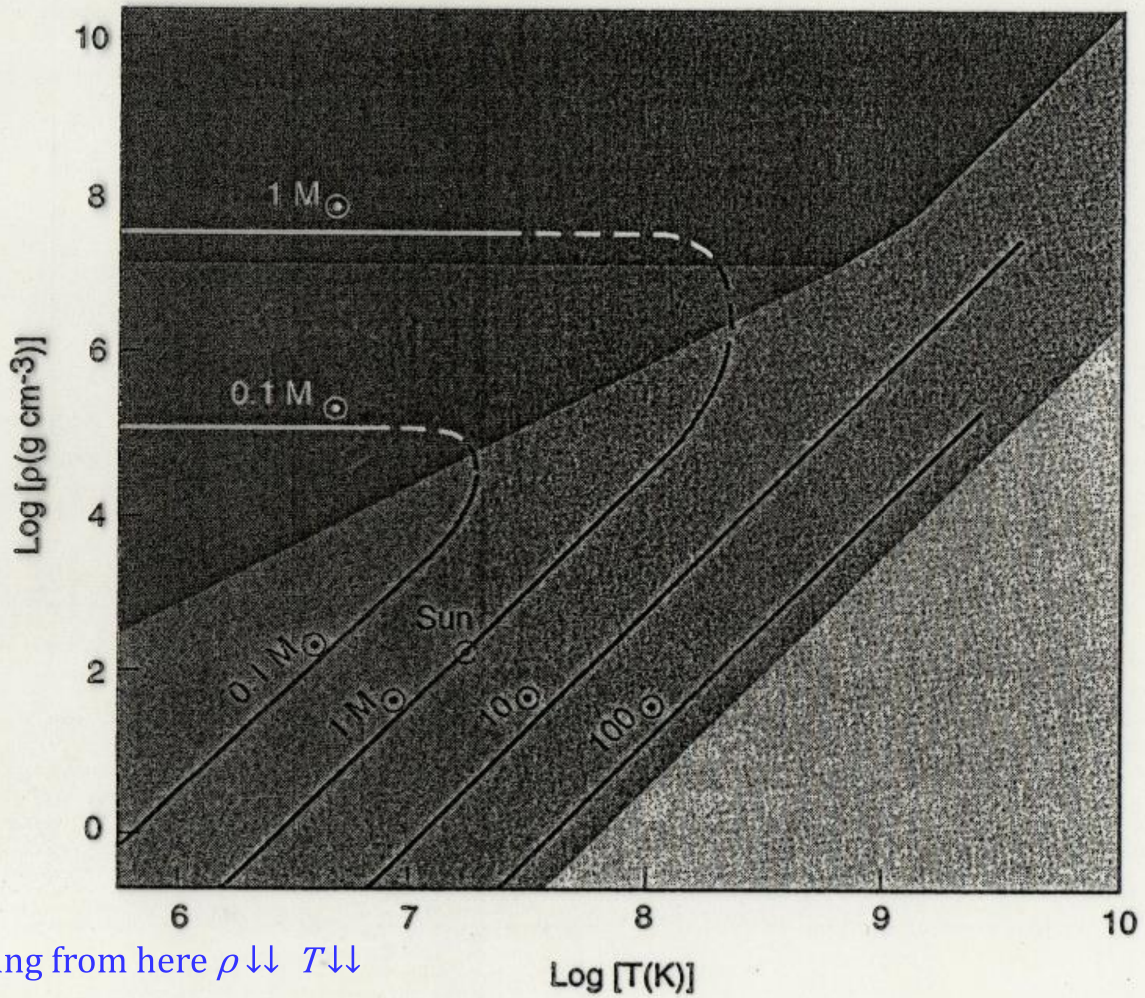


Figure 7.3 Outline of the stable and unstable zones in the temperature-density diagram.

$\gamma > 4/3 \Rightarrow$ stability



$T_c \propto \mu GM/R$

$\rho_c \propto T_c^3 \,/\, M^2$

so for a given M, slope =3

Starting from here $\rho\downarrow\downarrow$ $T\downarrow\downarrow$

Figure 7.4 Relation of central density to central temperature for stars of different masses within the stable ideal gas and degenerate gas zones.

From Prialnik

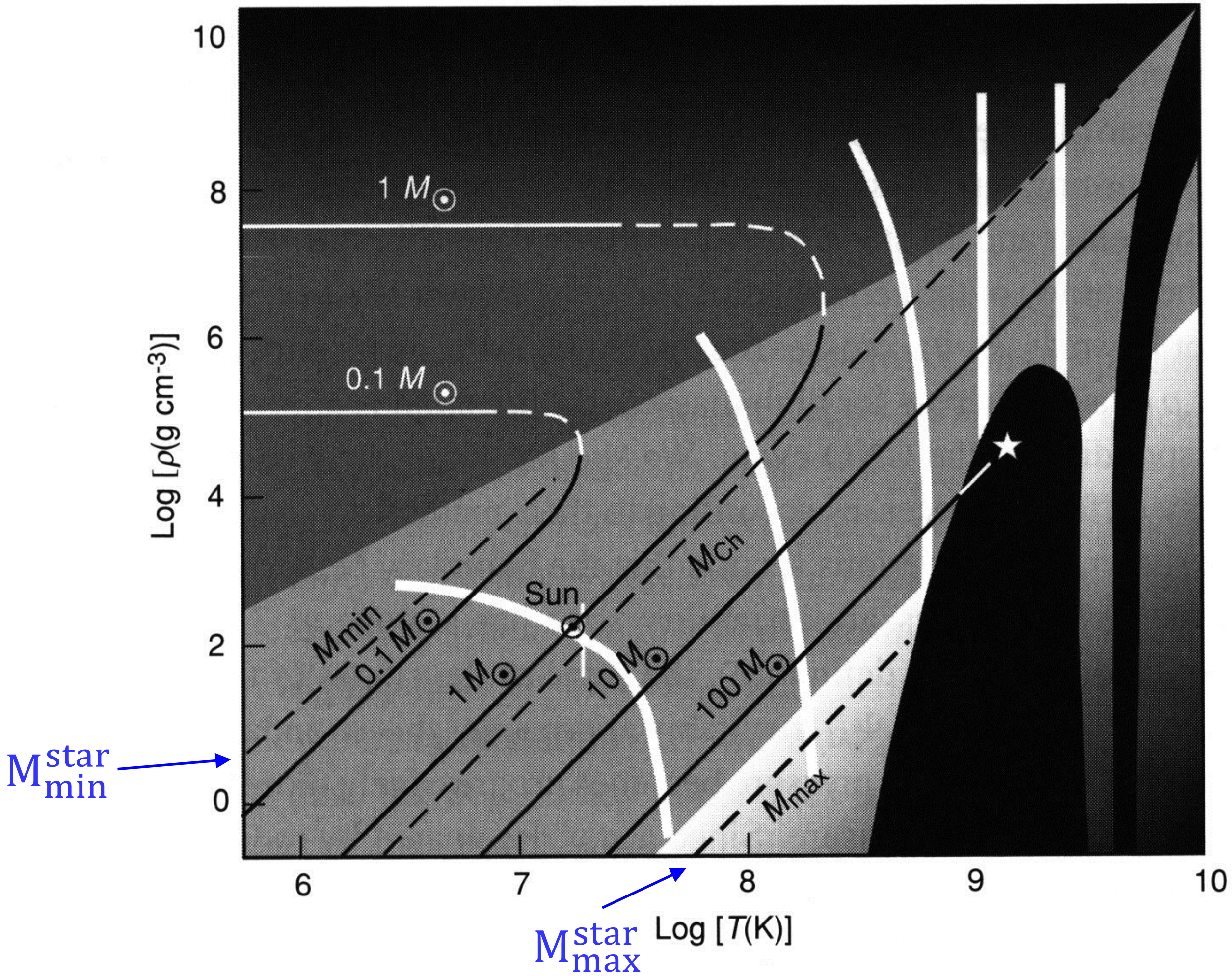


Figure 7.5 Schematic illustration of the evolution of stars according to their central temperature-density tracks.

From nonrelativistic to relativistic degeneracy

In a completely degenerate gas, the equation of state

$$\mathbb{P} \sim \rho^{5/3} \quad \text{NR}$$

<u>or</u>

$$\mathbb{P} \sim \rho^{4/3} \quad \text{ER}$$

cf. ideal gas

$$\mathbb{P} \sim \rho T$$

Hydrostatic equilibrium requires

$$\mathbb{P} \sim \frac{M^2}{R^4}$$

In the nonrelativistic case

There is a solution in case of NR.

$$P \sim \boxed{\frac{M^2}{R^4}} \sim \rho^{5/3} \sim \left(\frac{M}{R^3}\right)^{5/3} \sim \boxed{\frac{M^{5/3}}{R^5}}$$

$$\Rightarrow \quad \boxed{R \sim M^{-1/3}}$$

$\therefore$ R ↓ as M ↑ for WDs

The more massive of a WD, the smaller of its size.

Numerically

$$\log\left(\frac{R}{R_\odot}\right) = -\frac{1}{3}\log\left(\frac{M}{M_\odot}\right) - \frac{5}{3}\log(\mu_e) - 1.397$$

(Lang) Vol. 1

For 1 $M_\odot$, $R = 0.0126\, R_\odot$

$\langle \rho \rangle \sim 7 \times 10^5 \text{ g cm}^{-3}$

What happens in the ER case?

Total kinetic energy

$$E_k = N_e \frac{p^2}{2m} \quad (NR)$$

$$\left(\begin{array}{l} \text{degeneracy } p \approx \Delta p \\ \text{and } \Delta p \Delta x \sim \hbar \\ n_e = \dfrac{N_e}{R^3}, \quad \Delta p \sim \dfrac{\hbar}{\Delta x} \sim \dfrac{\hbar}{n^{-1/3}} \end{array} \right)$$

$$E_k = \frac{N_e (\Delta p)^2}{2 m_e} = \frac{N_e^{5/3}}{2 m_e} \cdot \frac{\hbar^2}{R^2}$$

$$\frac{1}{\mu_e}$$

$$\left(N_e = \frac{M Z}{A m_H} \approx \frac{1}{2} \frac{M}{m_H} \right)$$

Virial theorem (Equipartion)

$$E_p = \left| \frac{GM^2}{R} \right| \simeq 2\,E_K \Rightarrow R \approx \frac{\hbar^2}{G m_e m_H^{5/3}} \cdot M^{-1/3}$$

Note: $M^{1/3} R \approx \text{const}$

$$\frac{R}{R_\odot} \approx \frac{1}{74}\left(\frac{M_\odot}{M}\right)^{1/3}$$

The luminosity $L = 4\pi R^2 \sigma T_{\text{eff}}^4 \approx \frac{1}{74^2}\left(\frac{M_\odot}{M}\right)^{2/3}\left(\frac{T_{\text{eff}}}{6000}\right)^4 \quad [L_\odot]$

So a WD with $M = 0.4\ \mathrm{M}_\odot$ and $T_{\text{eff}} = 10^4$ K has $L = 3 \times 10^{-3}\ \mathrm{L}_\odot$

Gravity

$$g = \frac{GM}{R^2} \approx 74^2 \left(\frac{M}{M_\odot}\right)^{5/3} \frac{GM_\odot}{R_\odot^2}$$

For a WD with $M = 0.4\ \mathrm{M}_\odot$, $\mathrm{g} = 4 \times 10^7 \mathrm{cm\ s}^{-2}$

Gravitational Red shift

$$\frac{\Delta\lambda}{\lambda} = \left(1 - \frac{2GM}{Rc^2}\right)^{-1/2} \approx \frac{GM}{Rc^2} \approx 74 \left(\frac{M}{M_\odot}\right)^{4/3} \frac{GM_\odot}{R_\odot c^2}$$

In case of ER, $\breve{E}_R = N_e \, p c$

There is no solution in case of ER.

$$\breve{E}_R = N_e \frac{\hbar N_e^{1/3}}{R} \cdot c = \frac{M^{4/3} \hbar c}{m_H^{4/3} \cdot R}$$

$$\breve{E}_P = \left| \frac{GM^2}{R} \right|$$

$\breve{E}_R \simeq \breve{E}_P$, R cancels out; no solution for

$$M \equiv M(R)$$

$$P = \frac{M^2}{R^4} \text{ (if) } = \rho^{4/3} = \left(\frac{M}{R^3}\right)^{4/3} \Rightarrow \text{no solution}$$

- For degenerate gas, $M_{\mathrm{WD}}\ \uparrow$, $R_{\mathrm{WD}}\ \downarrow$
- For $M_{\mathrm{WD}} = 1\ M_{\odot}$, $R_{\mathrm{WD}} = 0.02\ R_{\odot}$
- There is an upper limit to the mass

$$M_{\mathrm{limit}} \approx \left(\frac{\hbar c}{G M_H^{4/3}}\right)^{3/2} \approx 2\ M_{\odot}$$

$$\mu_e = 1 \text{ (for H)}$$
$$= 2 \text{ (for He)}$$
$$= 56/26 = 2.15$$

Rigorously,

$$\boxed{M_{\mathrm{limit}} \approx \frac{5.836}{\mu_e^2} M_{\odot}}$$

$$\boldsymbol{M}_{\mathrm{limit}}\ (\mathrm{Fe}) = 1.26\ \mathrm{M}_{\odot}$$

Weinberg (1972) $M_{\mathrm{limit}} \approx 1.2\ M_{\odot}$, Later value $M_{\mathrm{limit}} \approx 1.44\ M_{\odot}$

TABLE 8.5. Central Densities, Total Mass, and Radius of Different White Dwarf Models, Taking $\mu_e = 2$ (Negligible Hydrogen Concentration).[a]

$M_{ch} = 1.44\, M_\odot$ needs corrections

— grav force on nuclei
deg. force on electrons
$\Rightarrow$ separation $\rightarrow \vec{E}$!

— e^- into nuclei $\Rightarrow n_e \downarrow$

$\log \rho_c$	$M/M_\odot$	$\log R/R_\odot$
5.39	0.22	−1.70
6.03	0.40	−1.81
6.29	0.50	−1.86
6.56	0.61	−1.91
6.85	0.74	−1.96
7.20	0.88	−2.03
7.72	1.08	−2.15
8.21	1.22	−2.26
8.83	1.33	−2.41
9.29	1.38	−2.53
∞	1.44	$-\infty$

[a] See text for comments (after M. Schwarzschild (Sc58b)). From *Structure and Evolution of the Stars* ©1958 by Princeton University Press, p. 232.

$$L = \sigma T_e^4 (4\pi R^2)$$

$$\log\left(\frac{L}{L_\odot}\right) = 4 \log\left(\frac{T_e}{T_{e\odot}}\right) + 2 \log\left(\frac{R}{R_\odot}\right)$$

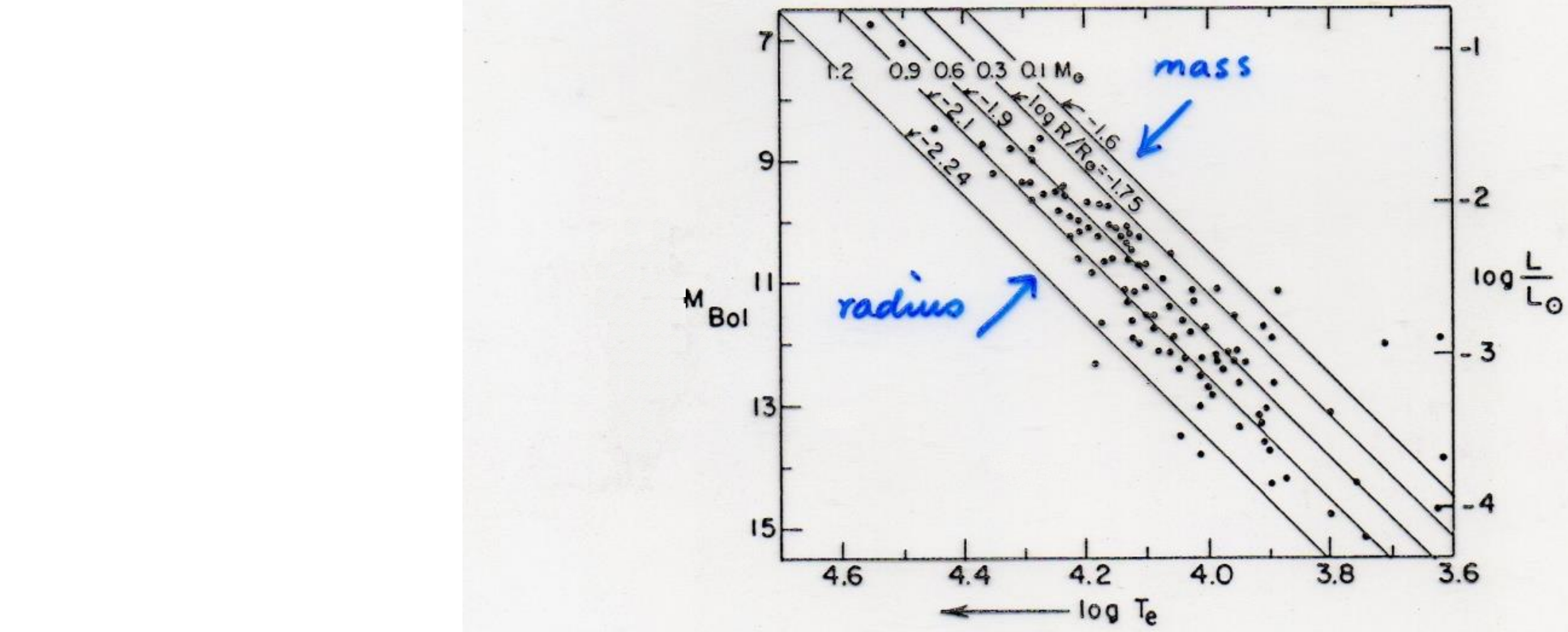
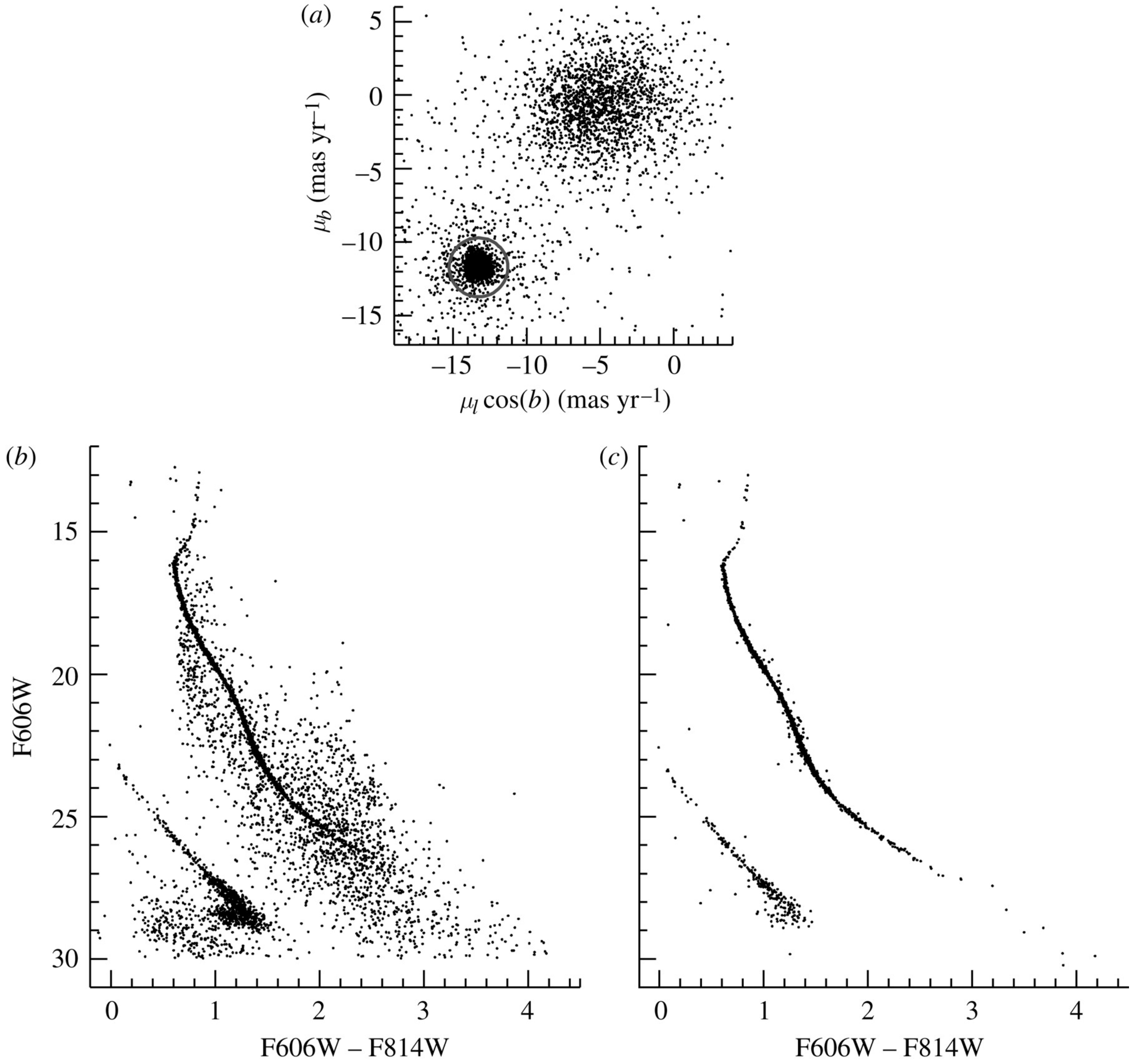


FIGURE 8.14. White dwarf Hertzsprung–Russell diagram. Lines of constant radius are shown. Also shown are the masses based on completely degenerate core models containing elements having $\mu_e = 2$ (after Weidemann (We68)). Reprinted with permission from *Annual Review of Astronomy and Astrophysics*, Vol. 6, ©1968 by Annual Reviews, Inc.).



Kalirai 2010

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doi:10.1088/0004-6256/143/1/11

A DEEP, WIDE-FIELD, AND PANCHROMATIC VIEW OF 47 Tuc AND THE SMC WITH *HST*: OBSERVATIONS AND DATA ANALYSIS METHODS*

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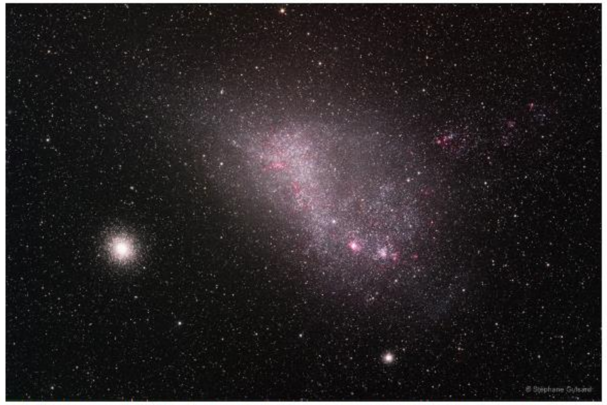


Figure 1. Wide-field ground-based image of the Small Magellanic Cloud (SMC) in the southern skies reveals two foreground Milky Way globular clusters, NGC 362 just below the SMC and 47 Tuc to the left of the galaxy. Although the main body of the SMC is separated from 47 Tuc by more than 2°, a diffuse stellar population persists to greater radii and represents a background source of stars in our study (as demonstrated later). This image subtends 6°8 × 4°5 and was taken with a 300 mm lens in 2007 September. The image was made by combining multiple 10 minute exposures in five visible filters (including Hα). Image is courtesy of Stéphane Guisard and reproduced here with permission, http://www.astrosurf.com/sguisard.
(A color version of this figure is available in the online journal.)

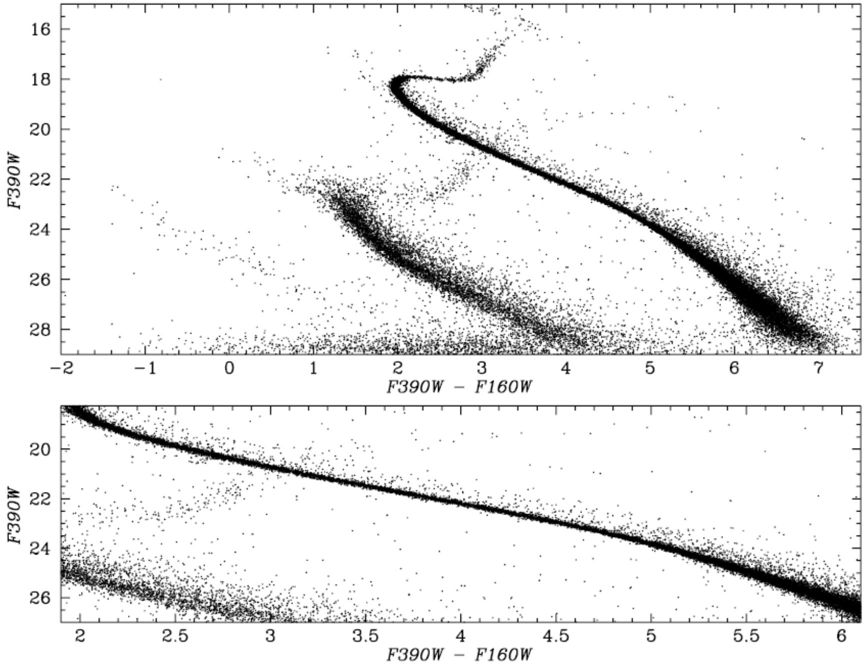
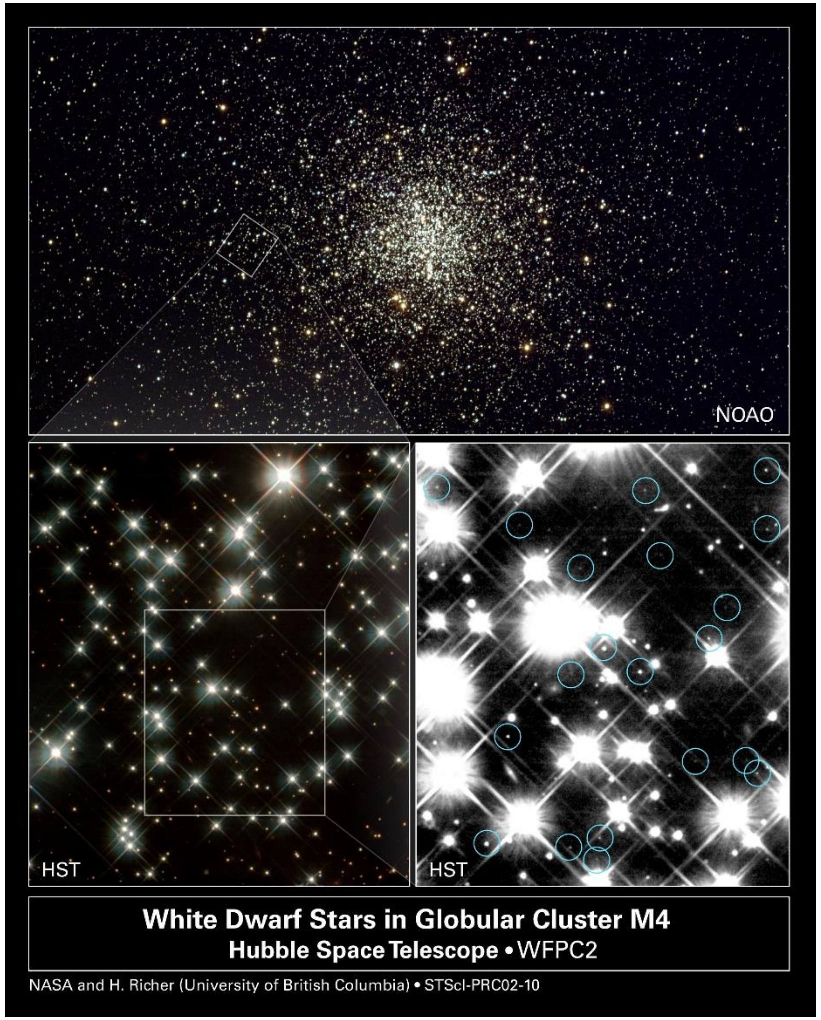


Figure 12. Panchromatic nature of this study is highlighted by constructing a CMD of the stellar populations over the widest baseline of $F390W - F160W$ (i.e., 0.4–1.7 μm). The combined WFC3/UVIS and IR data stretch the stellar populations over a color range of >9 mag (top panel). Despite their faintness in the IR, over 150 white dwarfs form a cooling sequence on this CMD. The bottom panel focuses on the main sequence of 47 Tuc, which is stretched over >4 mag of color.

White Dwarf Cooling

- WDs are supported by electron degeneracy pressure. With no sustaining energy source (such as fusion), they continue to cool and fade ➔ very faint
- The luminosity of the faintest WDs in a star cluster ←➔ cooling theory ➔ age
- The age of the oldest globular cluster = lower limit of the age of the universe

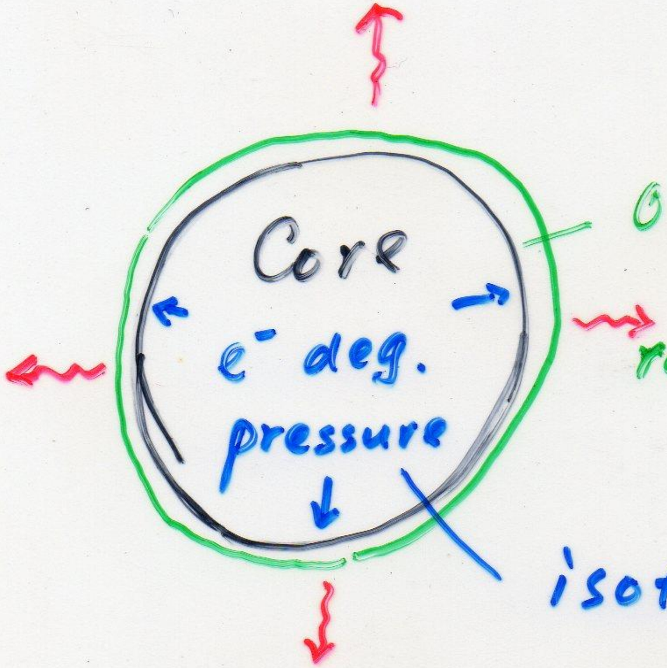


Limiting V=30

White Dwarf Cooling

Evolution of a White Dwarf

Mestel 1952



outer layer, $\rho \to 0$, $T \to 0$

radiation, energy from thermal energy of ions

isothermal

Degenerate gas ≈ metal ; v. good conductor

⇒ isothermal core

ON THE THEORY OF WHITE DWARF STARS

I. The Energy Sources of White Dwarfs

L. Mestel

(Communicated by F. Hoyle)

(Received 1952 May 9)

Summary

Present theories of the origin of white dwarfs are discussed; it is shown that all theories imply that there can be no effective energy sources present in a white dwarf at the time of its birth. The temperature distribution of a white dwarf is then discussed on the assumption that no energy liberation occurs within the star, and that it radiates at the expense of the thermal energy of the heavy particles present. In the resulting picture, a white dwarf consists of a degenerate core containing the bulk of the mass, surrounded by a thin, non-degenerate envelope. The energy flow in the core is due to the large conductivity of the degenerate electrons, while the high opacity of the outer layer keeps down the luminosity to a low level. Estimates of the ages of observed white dwarfs are given and interpreted. Finally, it is shown that white dwarfs may accrete energy sources and yet continue to cool off, provided the temperature at the time of accretion is not too high; this suggests a possible model for Sirius B.

Boundary between the degenerate core & the radiative envelope (r_b)

$r < r_b$, $T = T_c$

$r > r_b$, $L = \text{const}$

$m(r > r_b) \simeq M$

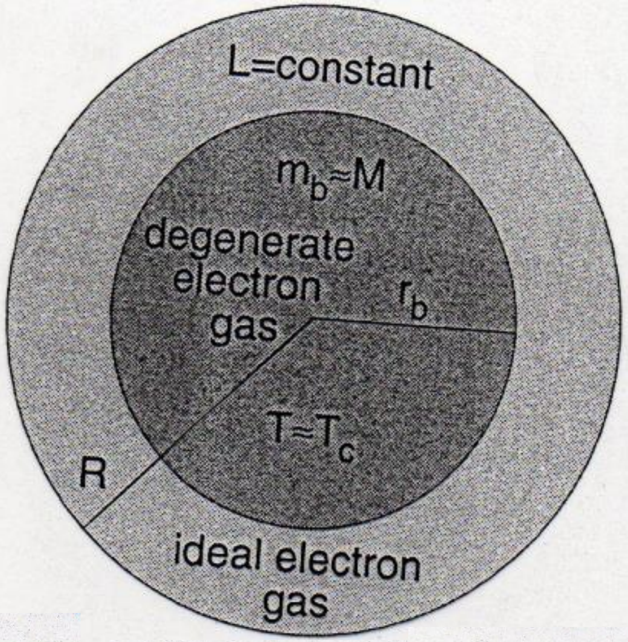


Figure 8.13 Sketch of the configuration of a cooling white dwarf.

In the envelope,

① $$\frac{dP}{dr} = -\rho \frac{GM}{r^2}$$ (i.e. $M(r) \rightarrow M$)

② $$\frac{dT}{dr} = -\frac{3}{4ac} \frac{\kappa \rho}{T^3} \frac{L}{4\pi r^2}$$ (i.e. $F(r) \rightarrow L$)

③ $$\kappa = \kappa_0 \underset{\sim}{\rho} T^{-3.5} = \kappa_0 \underbrace{\frac{\mu m_H}{k} P}_{} T^{-4.5}$$

Ideal gas

③ into ②, and ①/②

>

$$\frac{dP}{dT} = \frac{GM\,16\pi ac}{3\kappa L}T^3 = \frac{16\pi acGMT^3}{3\kappa_0\mu m_H P T^{-4.5}}\cdot\frac{k}{L}$$

$$= \frac{16}{3}K_1\frac{M}{LP}T^{+7.5}$$

$$P\,dP = \frac{16}{3}K_1\frac{M}{L}T^{7.5}dT$$

$\leftarrow$ integrate inward,

$$\frac{1}{2}P^2 = \frac{16}{3}K_1\frac{M}{L}\frac{T^{8.5}}{8.5}$$

$T\to 0,\ P\to 0$ as surface

In the envelope,

① $\frac{dP}{dr} = -\rho\frac{GM}{r^2}$ (i.e. $M(r)\to M$)

② $\frac{dT}{dr} = -\frac{3}{4ac}\frac{\kappa\rho}{T^3}\frac{L}{4\pi r^2}$ (i.e. $F(r)\to L$)

③ $\kappa = \kappa_0\underset{\sim}{\rho}T^{-3.5} = \kappa_0\underbrace{\frac{\mu m_H}{k}}P\,T^{-4.5}$

$$P = \cdots \left(\frac{M}{L} T^{8.5}\right)^{1/2}$$

$$\boxed{\frac{1}{2} P^2 = \frac{16}{3} K_1 \frac{M}{L} \frac{T^{8.5}}{8.5}}$$

$$P(T) = \left(\frac{64}{51} K_1\right)^{1/2} \left(\frac{M}{L}\right)^{1/2} T^{17/4}$$

This is the general radiative zero solution to the outer envelope (atmosphere) of stars

or

(4) $$\rho(T) = K_2 \left(\frac{M}{L}\right)^{1/2} T^{13/4}$$

At r_b, e^- ideal gas pressure = degenerate gas pressure

$$P_e = \left(\frac{k}{\mu m_H}\rho T\right)_b = P_{deg} = K_1' \left(\frac{\rho}{\mu_e}\right)_b^{5/3}$$

$$\rho T = K_2' \rho^{5/3}$$

$$\rho = K_3' T_b^{3/2}$$ A

Here $T_b = T_c$

$$(4) \quad \rho(T) = K_2 \left(\frac{M}{L}\right)^{1/2} T^{13/4}$$

$$\therefore \frac{L}{M} \sim \frac{T^{13/2}}{\rho^2} \sim \frac{T^{13/2}}{T^3} \sim T_c^{3.5}$$

$$\boxed{\frac{L}{M} = K T_c^{3.5}}$$

$L \leftarrow\rightarrow T_c$

$$\frac{L}{L_\odot} = 6.4 \times 10^{-3} \frac{\mu}{\mu_e^2} \frac{M}{M_\odot} \frac{1}{\kappa_0} T_c^{3.5}$$ ←→ chemical composition and opacity

Numerically, with constants (μ, μ_e, k_3') typical for a WD

$$\frac{L/L_\odot}{M/M_\odot} \simeq 6.8 \times 10^{-3} \left(\frac{T_c}{10^7 K} \right)^{3.5}$$

cf $T_F \sim 10^9 K$

or

$$\boxed{T_c \simeq 4 \times 10^7 \left(\frac{L/L_\odot}{M/M_\odot} \right)^{2/7} [K]}$$

B

The interior of a WD need not be exceedingly hot.

Energy source: $E_{\text{thermal}}^{\text{ions}} = (3/2)\,\dfrac{M}{\mu_I\, m_H}\, kT$

Luminosity $L = -d\, E_{\text{thermal}}^{\text{ions}}/dt$

$$= -(3/2)\,\frac{M}{\mu_I\, m_H}\, k\, \boxed{dT_c/dt}$$

$$\frac{L}{M} = K\, T_c^{3.5}$$

$$\frac{dL}{dt} = KM\,\frac{7}{2}\,T_c^{5/2}\,\boxed{\frac{dT_c}{dt}}$$

⑤ $\therefore\ L = -\dfrac{3}{7}\,\dfrac{M}{\mu m_H}\,k\,\dfrac{T_c}{L}\,\dfrac{dL}{dt}$

$$\Rightarrow\ \boxed{\frac{dL}{dt} = -M\,T_c^{6}}$$

Cooling rate ↓↓↓ as T_c ↓

$$L = -\frac{M\,T_c}{L}\,\frac{dL}{dt}$$

$$\frac{dL}{dt} = -\frac{L^2}{MT_c} = \frac{M^2}{MT_c}\,T_c{}^{7}$$

Thermal energy of ions in the isothermal core

= energy source of a white dwarf

$$E_{K,ion} = \frac{3}{2} \frac{M}{\mu_I m_H} k T_c$$

Luminosity $$L = -\frac{dE_K}{dt} = -\frac{3}{2} \frac{M}{\mu_I m_H} k \frac{dT_c}{dt}$$

$L \downarrow$ as $T_c \downarrow$

but $T_c \sim L^{2/7}$

⇒ lower-mass WD, evolves slowlier

Cooling timescale, from T_c', L' to T_c, L

Integrate (5)

$$\tau_{cool} = 0.6 \frac{k}{\mu_I m_H} M \left(\frac{T_c}{L} - \frac{T_c'}{L'} \right)$$

If $T_c' \gg T_c$ $\left(\frac{T_c'}{L'} \sim T_c'^{\,-2.5} \right) \Rightarrow \frac{T_c}{L} \gg \frac{T_c'}{L'}$

$$\boxed{\tau_{cool} \approx 2.5 \times 10^6 \left(\frac{M/M_\odot}{L/L_\odot} \right)^{5/7}}$$ [yr]

Core Temperature

$M \approx \mathrm{M}_\odot,\ {}^{L}/_{L_\odot} \approx 10^{-4} - 10^{-2}$ B $\rightarrow T_\mathrm{c} \approx 10^6$ K

A $\rightarrow \rho_\mathrm{b} \approx 10^3$ g cm^{-3}

Envelope

$$\ell \approx \frac{P}{\rho g} \approx \frac{kT}{\mu g}$$

$T \sim 10^6$ K, $l \approx 1 - 10$ km

Envelope mass $< 4\pi R^2 l\, \rho_\mathrm{b} \approx 2 \times 10^{-4}\ \mathrm{M}_\odot$, is indeed small

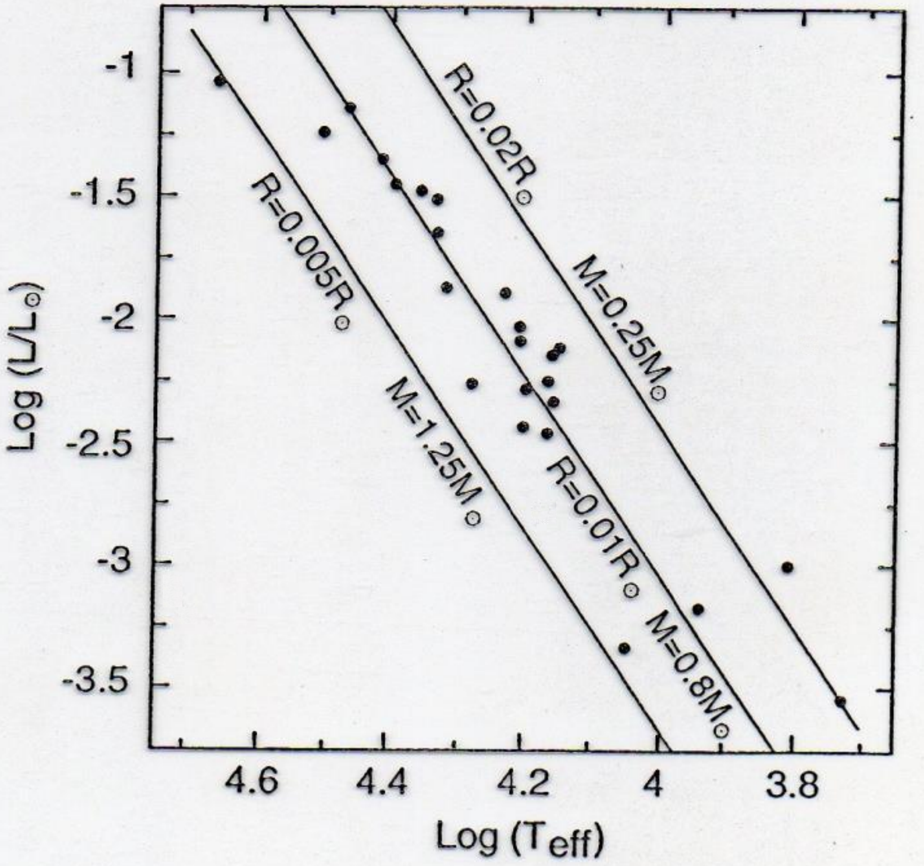


Figure 8.15 White dwarfs in the H–R diagram. Lines of constant radius (mass) are marked [data from M. A. Sweeney (1976), *Astron. & Astrophys.*, 49].

$MR^3 = \text{const}$, and $L \propto R^2\,\mathrm{T_{eff}}$

→ WD evolutionary tracks

$$\log\left(\frac{L}{L_\odot}\right) = 4\log\left(\frac{\mathrm{T_{eff}}}{T_\odot}\right) - \frac{2}{3}\log\left(\frac{M}{M_\odot}\right) + \mathrm{C}$$

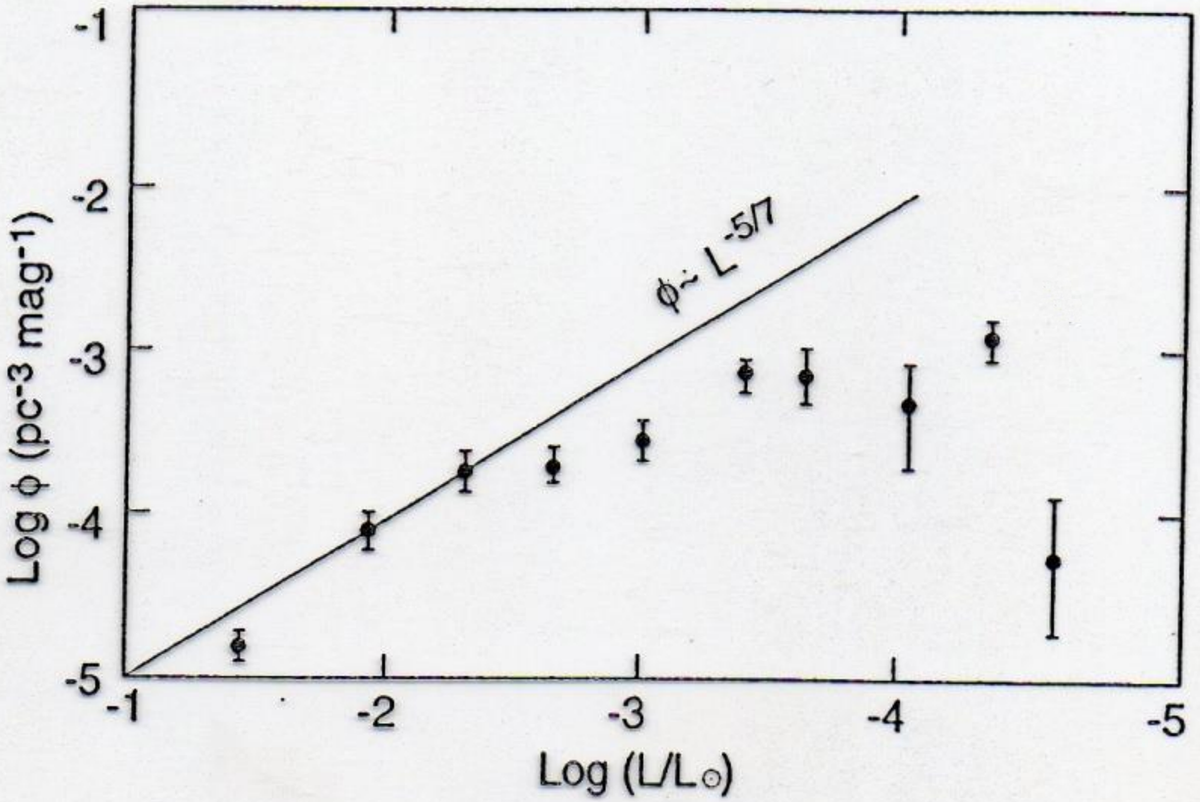


Figure 8.14 White dwarf luminosity function: number density of white dwarfs within a logarithmic luminosity interval corresponding to a factor of $10^{2/5} \approx 2.5$ against luminosity [data from D. E. Winget et al. (1987), *Astrophys. J.*, 315].

The Astrophysical Journal, 574:L155–L158, 2002 August 1

THE WHITE DWARF COOLING SEQUENCE OF THE GLOBULAR CLUSTER MESSIER 4[1]

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Received 2002 March 5; accepted 2002 June 26; published 2002 July 11

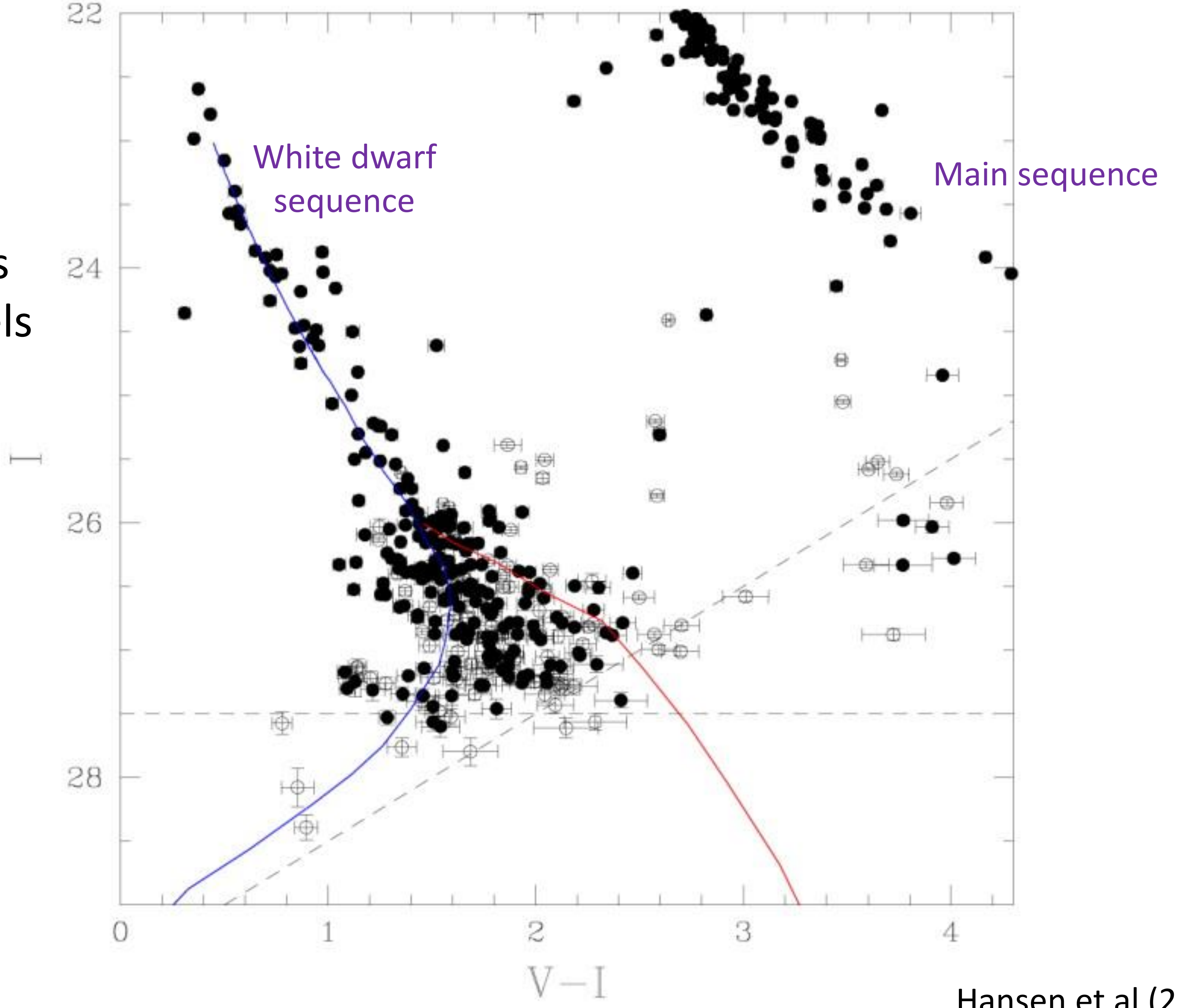
ABSTRACT

We present the white dwarf sequence of the globular cluster M4, based on a 123 orbit *Hubble Space Telescope* exposure, with a limiting magnitude of $V \sim 30$ and $I \sim 28$. The white dwarf luminosity function rises sharply for $I > 25.5$, consistent with the behavior expected for a burst population. The white dwarfs of M4 extend to approximately 2.5 mag fainter than the peak of the local Galactic disk white dwarf luminosity function. This demonstrates a clear and significant age difference between the Galactic disk and the halo globular cluster M4. Using the same standard white dwarf models to fit each luminosity function yields ages of 7.3 ± 1.5 Gyr for the disk and 12.7 ± 0.7 Gyr for M4 (2 σ statistical errors).

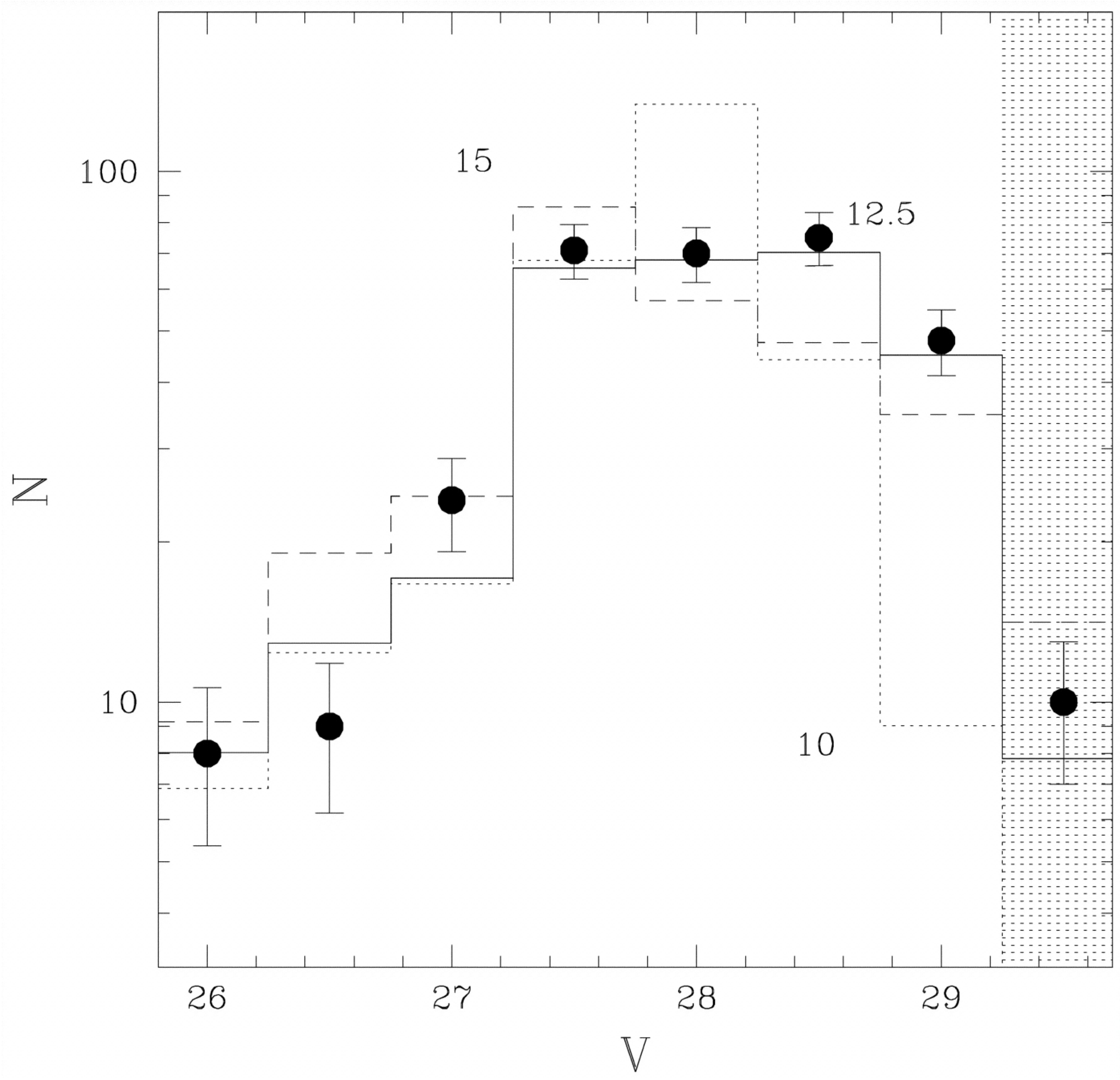
White dwarf sequence of M4

Blue – H atmosphere models
Red – He atmosphere models

for a 0.6 M" WD



Hansen et al (2002)



The observed luminosity function of the white dwarfs in M4 (after correction of incompleteness)

versus

model predictions for different ages

Hansen et al (2002)

- The WD envelope is typically thin, ~1% of the total WD radius.
- DA WD: layer of $M_{\mathrm{He}} \sim 10^{-2}\ M_{\mathrm{WD}}$ outside the CO core, then an outer layer $M_{\mathrm{H}} \sim 10^{-4}\ M_{\mathrm{WD}}$
- A non-DA WD layer of $M_{\mathrm{He}} \sim 10^{-2} - 10^{-3}\ M_{\mathrm{WD}}$