THE SCHOENBERG–CHANDRASEKHAR LIMIT: A POLYTROPIC APPROXIMATION

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(Received 29 February, 1988)

Abstract. The existence of a maximum isothermal core mass fraction (q_{max}), the Schoenberg–Chandrasekhar limit, is one of the 'classic' results from the theory of stellar structure. This limit can be demonstrated through a simplified composite polytrope model in which an isothermal core is surrounded by an $n = 1$ polytrope envelope. While this model underestimates q_{max} by $\sim 25\%$ in the homogeneous case, it is accurate to within 5% in the more realistic inhomogeneous situation.

1. Introduction

With the exhaustion of hydrogen in its central regions a star evolves a three part structure: a helium rich core, a thin hydrogen burning shell source, and an outer envelope which has retained the original, zero-age, chemical composition. That the central helium rich core may become isothermal was first suggested by Gamow (1938). This can be demonstrated through the simplified argument: since $\mathrm{d}L/\mathrm{d}r \sim \varepsilon(r)$ and in the core $\varepsilon = 0$ and $L(r = 0) = 0$ it follows that $L(r \leq r_c) = 0$ as well, where r_c is the core radius. Furthermore, if we consider the temperature gradient $\mathrm{d}T/\mathrm{d}r \sim L(r)$ then $T(r < r_c) = \text{constant} = T_c$. That is the core is isothermal with temperature T_c. Detailed numerical calculations have shown, however, that this approximate situation only occurs in a restricted mass range. During the hydrogen shell-burning phase calculations show that the helium core not only increases in mass, but begins to contract. If the core is isothermal contraction requires that the density increases in order to provide the pressure support for the overlying layers. Two mass-dependent factors, however, come into play (Cox and Giuli, 1968). For $M \lesssim 1.5\,M_\odot$ the isothermal core eventually becomes degenerate, with the degeneracy contributing to the pressure support. For $M \gtrsim 6.0\,M_\odot$ the core at hydrogen exhaustion is sufficiently large that with contraction there is a significant release of gravitational energy which results in rapid core heating and isothermality is in fact never achieved. Only in the mass range $1.5 \lesssim M/M_\odot \lesssim 6.0$ then is it expected that an isothermal, non-degenerate, helium-rich core will develop. Having attained this situation, however, it transpires that a critical core mass fraction $q_{max} = M_{c,\,max}/M_*$ exists, where $M_{c,\,max}$ is the maximum core mass and M_* is the total stellar mass. In this manner as soon as $q > q_{max}$ the core can no longer support the weight of overlying layers and, in consequence, it is forced to rapidly contract. This contraction leads to core heating and the establishment of a non-zero temperature gradient. Once T_c reaches $1\text{–}2 \times 10^8$ K central helium burning will begin. At this point the star will be at the tip of its red giant branch in the HR diagram. It is this maximum

Astrophysics and Space Science **147** (1988) 219–227.

central core mass $M_{c,\,\max}$ commonly referred to as the Schoenberg–Chandrasekhar limit, that we consider in the remainder of this article.

2. First Investigations

The first specific investigation of stellar models with isothermal cores was undertaken by Henrich and Chandrasekhar (1941). In this analysis only homogeneous models were considered, that is the core and the envelope were assumed to have the same chemical composition. Specifically, two configurations were investigated: (i) isothermal core plus radiative, $n = 3$ polytrope, envelope, and (ii) isothermal core with a radiative point source envelope in which Kramers opacity law was taken to operate. In both cases they discovered that an upper limit to the total mass that could be contained in the isothermal core existed. For mmodels of type (i) they found $q_{\max} \simeq 0.38$, while for models of type (ii) they found a slightly lower value of $q_{\max} \simeq 0.35$. In a subsequent analysis, Schoenberg and Chandrasekhar (1942) considered the more realistic chemically inhomogeneous configurations. These models, similar in construction to those of type (i) in Henrich and Chandrasekhar, had core and envelope compositions appropriate to that of a hydrogen exhausted core and a hydrogen rich envelope. In terms of the mean molecular weights this can be expressed $\mu_c \simeq 2.0\mu_e$, where the c and e subscripts correspond to the core and envelope, respectively. With these conditions they found that the maximum core mass fraction was $q_{\max} \simeq 0.101$. This result demonstrated that the upper limit to the core mass fraction was a decreasing function of μ_c/μ_e.

That a critical isothermal core mass fraction should exist can be demonstrated through an application of the virial theorem. Generalizing the earlier results of McCrea (1957), Stein (1966) showed that there is a maximum pressure $P_{\max}$ that a non-degenerate isothermal core in hydrostatic equilibrium can support at its surface. If the core is to support the weight of the overlying envelope then the inequality $P_{\max} > P_{\text{env}}$ must hold, where P_{env} is the pressure at the base of the envelope. This condition can be cast in the form (Stein, 1966; Cox and Giuli, 1968)

$$q_{\max} = A\left(\frac{\mu_e}{\mu_c}\right)^2, \tag{1}$$

where A is constant. Henrich and Chandrasekhar's 1941 results, corresponding to $\mu_c = \mu_e$, imply that $A \simeq 0.37$. The constant A was calculated by Stein (1966) for the linear stellar model (in which it is assumed that $\rho(r) = \rho_c(1 - r/R_*)$). He found in this case that $A = 0.30$. In a more complex calculation, where the influence of core rotation was included in the virial equation, Maeder (1971) found, as a consistency check, that at rest $A = 0.38$. Below we try to recreate these results, particularly the derivation of the constant A, by appealing to a simplified composite polytropic model.

3. The Polytropic Model

To a first approximation the structure of a star in the mass range $1.5 \lesssim M/M_\odot \lesssim 6.0$ following core hydrogen exhaustion can be described as an isothermal core surrounded by a radiative envelope of different chemical composition. Such a configuration can be thought of as an $n = \infty$ core, $n = 3$ envelope, composite polytrope with a mean molecular weight jump at the core-envelope boundary. Since the boundary conditions at the base of the envelope are not those appropriate to the standard, and tabulated, solution for the $n = 3$ polytrope, it is necessary to numerically integrate the Lane–Emden equation for each new position of the core-envelope interface $\xi = \xi_i$ and for each new core-envelope composition difference μ_c/μ_e. Since we wish to consider many values of ξ_i and μ_c/μ_e. it would be an advantageous time saving if the envelope calculations could be reduced to a simple form. In an earlier communciation (Beech, 1988), we outlined the properties of the $n = 3$ core, $n = 1$ envelope composite polytrope. We once again adopt the $n = 1$ envelope approximation here. This allows us to calculate the envelope structure in a straightforward analytic fashion. Apart from the numerical simplification it affords there are no compelling physical arguments for the adoption of the $n = 1$ polytrope envelope. The approximation does, however, describe the important feature of attaching a finite mass, non-infinite-radius, envelope to an isothermal core.

In the isothermal core the pressure term at equilibrium is given by

$$P = k_c \rho + D \,, \tag{2}$$

where $k_c = RT_c/\mu_c$ and $D = aT_c^4/3$. If this is combined with the equations of hydrostatic equilibrium, the isothermal analogue to the Lane–Emden equation results (Chandrasekhar, 1939) such that

$$\frac{1}{\xi^2}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi^2 \frac{\mathrm{d}\psi}{\mathrm{d}\xi}\right) - e^{-\psi} = 0 \,; \tag{3}$$

with the boundary conditions

$$\psi = 0 \,, \qquad \frac{\mathrm{d}\psi}{\mathrm{d}\xi} = 0 \quad \text{at} \quad \xi = 0 \,, \tag{4}$$

Equation (3) combined with boundary conditions (4) can be integrated numerically, and in our analysis we use the extensive tabulations of Horedt (1986).

With the envelope approximated as an $n = 1$ polytrope the equilibrium model is described by the Lane–Emden equation

$$\frac{1}{\eta^2}\frac{\mathrm{d}}{\mathrm{d}\eta}\left(\eta^2 \frac{\mathrm{d}\phi}{\mathrm{d}\psi}\right) + \phi = 0 \,, \tag{5}$$

which has the general solution

$$\phi(\eta) = A \, \frac{\sin(\eta - B)}{\eta} \,, \tag{6}$$

where A and B are constants to be determined by the core-envelope boundary conditions. If $\phi(\eta_i)$, $\phi'(\eta_i)$, and η_i are specified at the base of the envelope, then

$$A = \phi(\eta_i)\eta_i \operatorname{cosec}(\eta_i - B) \tag{7}$$

and

$$B = \eta_i - \cot^{-1}\left\{\phi'(\eta_i) + \frac{1}{\eta_i}\right\}; \tag{8}$$

the 'surface' of the envelope is characterized by $\phi(\eta_s) = 0$, which from Equation (6) implies that

$$\eta_s = B + \pi . \tag{9}$$

4. The Core-Envelope Boundary

At the boundary between the core and envelope continuity of pressure (temperature), mass and radius are required to hold. Hence, with standard notation (cf. Chandrasekhar, 1939), if the core boundary is described by $\xi = \xi_i$ then,

$$k_c\rho_c e^{-\psi(\xi_i)} + D = k_e\rho_e^2\phi^2(\eta_i) \quad \text{(pressure terms)}, \tag{10}$$

$$4\pi\left\{\frac{k_c}{4\pi G}\right\}^{3/2}\rho_c^{-1/2}\left(\xi^2\frac{d\psi}{d\xi}\right)_{\xi_i} = 4\pi\left\{\frac{k_e}{2\pi G}\right\}^{3/2}\rho_e\left(-\eta^2\frac{d\phi}{d\eta}\right)_{\eta_i} \quad \text{(mass terms)}, \tag{11}$$

$$\left\{\frac{k_c}{4\pi G}\right\}^{1/2}\rho_c^{-1/2}\xi_i = \left\{\frac{k_e}{2\pi G}\right\}^{1/2}\eta_i \quad \text{(radial terms)}. \tag{12}$$

Since there is a jump in mean molecular weight across the core-envelope boundary and pressure and temperature are necessarily continuous there we require, in consequence, the continuity of ρ/μ. In this way,

$$\frac{\rho_c e^{-\psi(\xi_i)}}{\mu_c} = \frac{\rho_e}{\mu_e}\phi(\eta_i) \quad (\rho/\mu \text{ terms}). \tag{13}$$

Equations (10)–(13) offer four equations in the five unknowns k_e, ρ_e, $\phi(\eta_i)$, $\phi'(\eta_i)$, and η_i. To continue we choose the normalization,

$$\phi(\eta_i) = 1 . \tag{14}$$

Simple algebra then determines relations for the remaining terms: namely,

$$\eta_i = \frac{1}{\sqrt{2}}\left(\frac{k_c}{k_e}\right)^{1/2}\rho_c^{-1/2}\xi_i , \tag{15}$$

$$\rho_e = \rho_c\left(\frac{\mu_e}{\mu_c}\right)e^{-\psi(\xi_i)} , \tag{16}$$

$$\phi'(\eta_i) = -\frac{1}{\sqrt{2}}\left(\frac{k_c}{k_e}\right)^{1/2} \frac{e^{\psi(\xi_i)}}{\rho_c}\left(\frac{\mu_c}{\mu_e}\right)\left(\frac{d\psi}{d\xi}\right)_{\xi_i} \tag{17}$$

and

$$k_e = k_c\left(\frac{\mu_c}{\mu_e}\right)^2\left\{1 + \frac{D\, e^{\psi(\xi_i)}}{\rho_c k_c}\right\}\frac{e^{\psi(\xi_i)}}{\rho_c}\ . \tag{18}$$

Equations (6)–(9) combined with (15)–(18) give a complete description of the envelope, total mass, and configurational radius. Of interest to the investigation at hand is the description of the core mass fraction q. This can be expresed as

$$q(\xi_i) = \frac{M_c}{M_*} = \left(\frac{k_c}{2k_e}\right)^{3/2} \rho_c^{-3/2}\left(\frac{\mu_c}{\mu_e}\right) e^{\psi(\xi_i)}\ \frac{(\xi^2\, d\psi/d\xi)_{\xi_i}}{(-\eta^2\, d\phi/d\eta)_{\eta_s}}\ . \tag{19}$$

In order to completely determine a model we choose to specify ρ_c, T_c, μ_c, μ_e, and ξ_i.

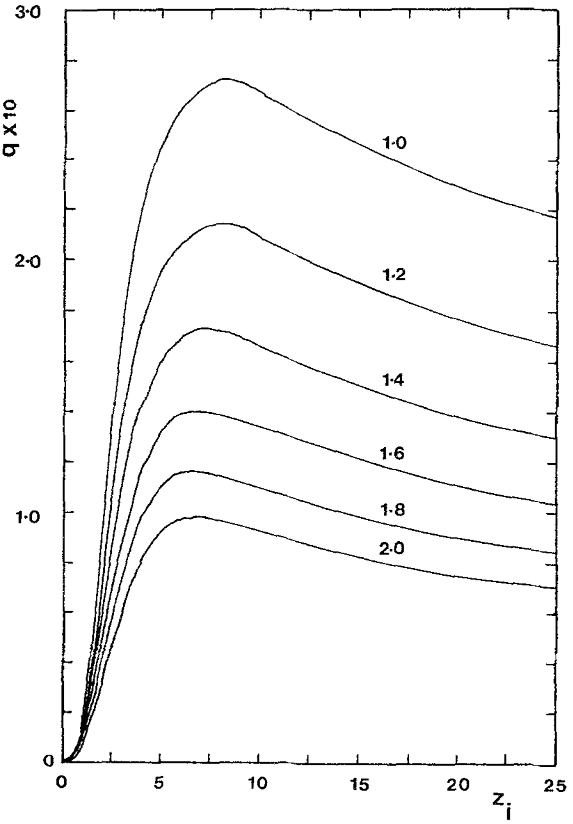

Fig. 1. The variation of core mass fraction q with core-envelope position ξ_i. Each curve is labelled by the choice of μ_c/μ_e.

5. Model Results

On the basis of detailed numerical calculations (see, e.g., Cox and Giuli, 1968) it is expected that the central density and temperature of the isothermal core will vary between $\sim 10^3 \lesssim \rho_c(\text{g cm}^{-3}) \lesssim \text{few} \times 10^4$ and $\sim 10^7 \lesssim T_c \lesssim 10^8$. For a given ρ_c, T_c combination and choice of μ_c/μ_e Equation (19) can be evaluated at various core-envelope positions $\xi = \xi_i$. Horedt (1986) has given an extensive tabulation of Equation (3) which we use to describe the isothermal core and core-envelope boundary conditions (15)–(18). With $\rho_c = 10^3$ g cm^{-3} and $T_c = 2 \times 10^7$ K the variation of core mass fraction with ξ_i is shown in Figure 1 for various μ_c/μ_e combinations. In Figure 2 the variation of the core mass fraction with the total model mass is shown when $\mu_c/\mu_e = 2$ and for various ρ_c, T_c combinations. From these two figures several properties of the composite polytrope model become apparent: for a given μ_c/μ_e there is a well-defined maximum core mass fraction $q_{\max}$ and that (as Figure 2 illustrates) this maximum is quite insensitive to the choice of central boundary conditions ρ_c and T_c. Using

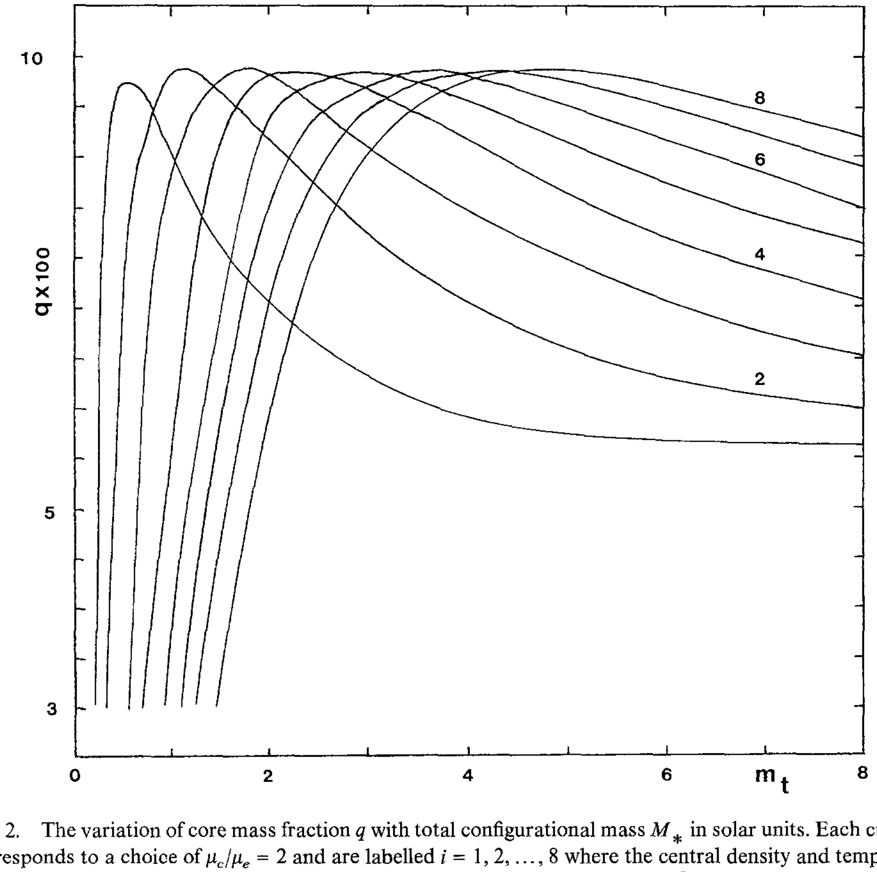

Fig. 2. The variation of core mass fraction q with total configurational mass M_* in solar units. Each curve corresponds to a choice of $\mu_c/\mu_e = 2$ and are labelled $i = 1, 2, \ldots, 8$ where the central density and temperature are given by $\rho_c = (2i) \times 10^3$ g cm^{-3} and $T_c = i \times 10^7$ K.

Figure 1 we can determine q_{max} for each μ_c/μ_e combination (see Table I). With Equation (1) as a guide a least-squares fit to q_{max} and $(\mu_e/\mu_c)^2$ from Table I gives

$$q_{max} = 0.23\left(\frac{\mu_e}{\mu_c}\right)^2 + 0.05\,, \tag{20}$$

with a goodness of fit coefficient $r = 0.9955$. A comparison between several evaluations of the constant in Equation (1) and the polytropic model result given in (20) is presented in Figure 3. The composite polytrope model appears to underestimate q_{max} by 25% for the homogeneous model ($\mu_c = \mu_e$), but is accurate to within 5% for the more realistic, inhomogeneous, model where $\mu_c = 2\mu_e$. For all the apparent crudeness of the polytropic approximation used in the envelope the composite model gives surprisingly accurate results.

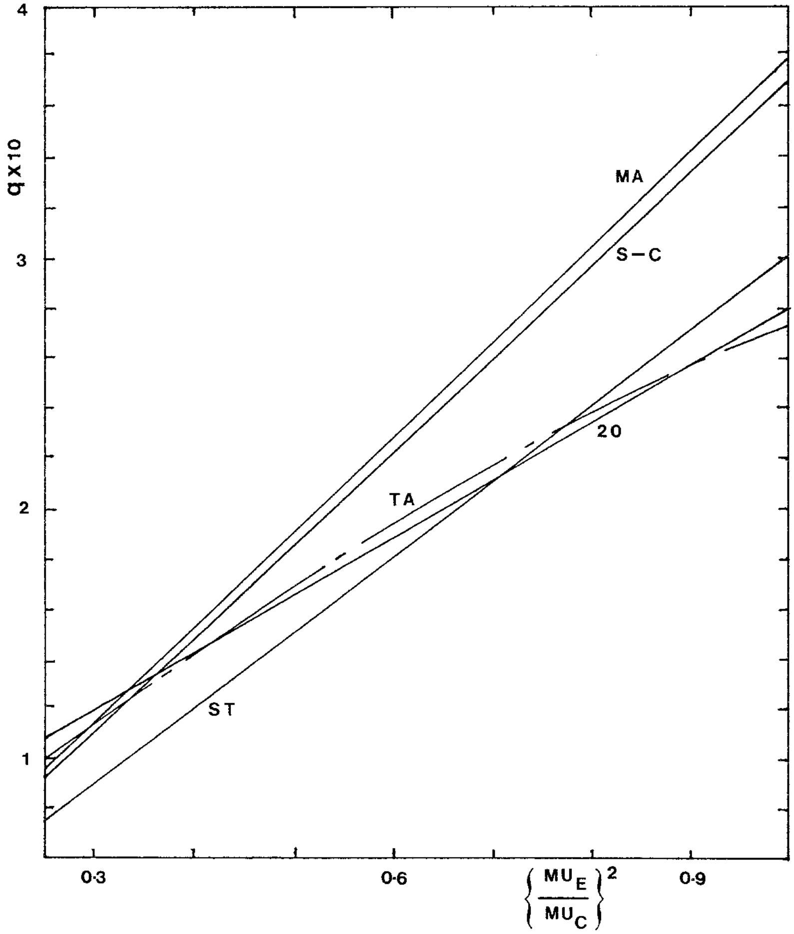

Fig. 3. A comparison between the various evaluations of Equation (1) and the composite polytrope results given in Equation (20). Key: Ma – Maeder (1971). S–C – Schoenberg and Chandrasekhar (1947). St. – Stein (1966). 20 – Equation (20). Ta – results given in Table I.

TABLE I

q_{max} values as derived from Figure 1

μ_c/μ_e	$(\mu_c/\mu_e)^2$	q_{max}
1.0	1.000	0.273
1.2	0.694	0.215
1.4	0.510	0.173
1.6	0.391	0.141
1.8	0.309	0.117
2.0	0.250	0.099

6. Discussion

Between core hydrogen exhaustion and the onset of core helium burning, a star undergoes dramatic internal reorganization and evolves to a completely new position in the HR diagram; the Main-Sequence dwarf becomes a cool red giant. It is clear that the Schoenberg–Chandrasekhar limit and/or the formation of an isothermal core are not the crucial mechanisms that turn a dwarf star into a giant (Eggleton and Faulkner, 1981). Rather, it is the development of a mean molecular weight gradient, increased central mass concentration and the change over from core to shell burning that are the important factors. It is unlikely that such structure can be successfully modelled as a single polytrope and the situation would seemingly suggest that at least a triple polytropic composite model is required. However, as Eggleton and Faulkner (1981), Yahil and Van den Horn (1985), and Beech (1986) have found, the tendency towards acquiring large radii can be demonstrated via the single polytrope approximation. The properties of red giant stars may possibly be describable through the so-called M-solutions to single polytropes (Chandrasekhar, 1939; Beech, 1986; Hjellming and Webbink, 1987), but to-date such models have not been fully developed. The model presented here is useful for describing the final and early post-Main-Sequence evolution of stars in the mass range $1.5 \lesssim M/M_\odot \lesssim 6.0$, and it also illustrated the existence of a maximum isothermal core mass fraction. Work in hand, however, is beginning to explore the properties of multiple polytropes beyond the simplified model presented here. To this end a mean molecular weight gradient between the core and envelope is explicitly included in the model. Preliminary results suggest that the evolution of the upper Main-Sequence stars can be well modelled in this way.

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