Stellar Interior

- A star has a stable configuration.
  - ✓ That is, there is a certain structure (mass distribution) to allow for such a force balance.

    Inward = gravity
    Outward = gas pressure (gradient)
    (ideal gas, degenerate gas)
    + magnetic pressure ($P_{\mathrm{mag}} = B^2/8\pi$)
    + radiation pressure ($P_{\mathrm{rad}} = 4\sigma T^4/3c$)
    + turbulence pressure ($P_{\mathrm{tur}} = \rho\, v^2/2$)

- How is the pressure sustained? Energy→ thermal pressure
  - ✓ How is the energy generated?
  - ✓ How is the energy transported?

Structure Equations

$$\frac{dm(r)}{dr} = 4\pi r^2 \rho(r)$$ Mass continuity (distribution)

$$\frac{dP(r)}{dr} = -\frac{g m(r) \rho(r)}{r^2}$$ Hydrostatic equilibrium

$$\frac{dL(r)}{dr} = 4\pi r^2 \rho(r)\, q(r)$$ Energy conservation

$$\frac{dT(r)}{dr} = -\frac{3\kappa\rho L(r)}{4ac\, 4\pi r^2 T^3}$$ by radiation

Energy transport

$$\frac{dT(r)}{dr} = \frac{\gamma - 1}{\gamma}\frac{T}{P}\frac{dP(r)}{dr}$$ by convection

$$P = P(\rho, T, \mu)$$

Equation of state

$$\kappa = \kappa(\rho, T, \mu)$$

Opacity

$$q = q(\rho, T, \mu)$$

Nuclear reaction rate

Boundary conditions: $m(r) \to 0$ and $L(r) \to 0$ as $r \to 0$
$T(r) \to 0$, $P(r) \to 0$, and $\rho(r) \to 0$ as $r \to R_*$

$Variables: m, r, \rho, T, P, \kappa, L, \mu, and\ q$

Vogt-Russell "theorem"

Given hydrostatic and thermal equilibrium with energy produced by nuclear reactions, the internal structure of a star, and its subsequent evolution, is uniquely determined by the mass and chemical composition of the star.

In fact, … by any two variables above, cf. the HRD. It is not really a "theorem" in the mathematical sense, i.e., not strictly valid. It is a "rule of thumb". There are other factors, too, such as magnetic field or rotation, though these usually have little effect.

The Poynting vector of an EM wave,

$$\boldsymbol{S} = \frac{\mathrm{c}}{4\pi}\, \boldsymbol{E} \times \boldsymbol{B}$$

Radiation pressure is 1/3 of the EM energy density

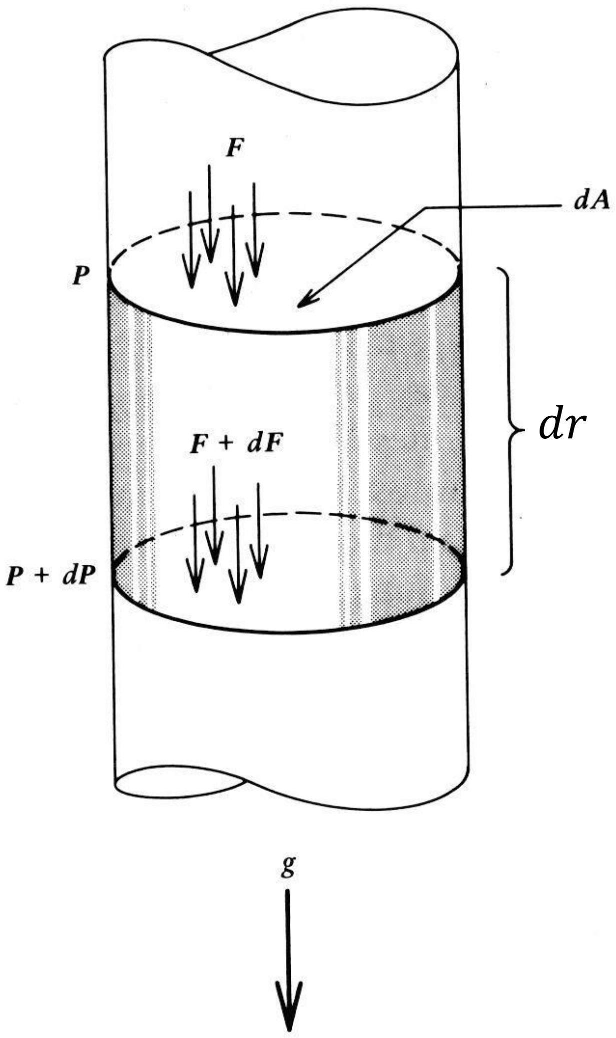
$$P = \frac{1}{3}\left[\frac{1}{8\pi}(\varepsilon E^2 + \mu B^2)\right]$$

The carrier wave velocity, the phase velocity $v = \frac{\omega}{k}$

The velocity of the modulation, the group velocity $u = \frac{\partial\omega}{\partial k}$;
this is the information (or energy) is transported.

Harwit

Hydrostatic equilibrium



In general, the equation of motion is

$$\ddot{r} = -\frac{Gm}{r^2} - \frac{1}{\rho}\frac{\partial P}{\partial r} = -\frac{Gm}{r^2} - 4\pi r^2 \frac{\partial P}{\partial m}$$

The LHS is usually null, unless there is free fall or explosion.

$$\text{Force} = \text{mass} \cdot \text{acceleration}$$

$$-dP\, dA = \rho(r)\, dA\, dr \cdot g(r)$$

$$\frac{dP}{dr} = -\rho(r)\, g(r) = -\rho(r)\frac{GM(r)}{r^2}$$

Hydrostatic equilibrium

$$\frac{dP(r)}{dr} = -\frac{G\,m(r)}{r^2}\rho(r) = -g(r)\,\rho(r) \ldots (1)$$

- ✓ $m(r)$: total mass inside radius r
- ✓ $P_{\text{total}} = P_{\text{gas}} + P_{\text{e}} + P_{\text{rad}}$

Mass continuity

$$\frac{dm(r)}{dr} = 4\pi r^2 \rho(\mathrm{r}) \ldots (2)$$

(1)/(2)

Using mass as the independent variable, $r = r(m)$, is preferred because mass is usually given and fixed (but r is not.)

$$(1) \longrightarrow \frac{dP(r)}{dm} = -\frac{Gm(r)}{4\pi r^4}$$

$$(2) \longrightarrow \frac{dr}{dm} = \frac{1}{4\pi r^2 \rho(r)}$$

Boundary conditions (1) at $m = 0, r = 0$,
(2) at $m = M$ or $r = R$, $P = 0$.

The stellar structure equations then become

$$\frac{dP(m)}{dm} = -\frac{G\,m}{4\pi r^4}$$

$$\frac{dr}{dm} = \frac{1}{4\pi r^2\,\rho}$$

$$\frac{dL(m)}{dm} = q(r)$$

$$\frac{dT(m)}{dm} = -\frac{3\kappa L(r)}{4ac\,(4\pi r^2)^2 T^3}$$

$$P = \frac{\rho}{\mu m_H} kT + P_e + \frac{1}{3}\,aT^4$$

$$\kappa = \kappa_0\,\rho^a\,T^b$$

$$q = q_0\,\rho^m\,T^n$$

Prianik

A polytropic (thermodynamic) process obeys

$$PV^{\alpha} = \text{const}$$

α is the polytropic index

- ✓ $\alpha = 0, P = \text{const}$ → isobaric process

For an ideal gas

- ✓ $\alpha = 1$ → isothermal process
- ✓ $\alpha = \gamma = c_p/c_v$ → isentropic (= adiabatic and reversible) process
- ✓ $\alpha \to \infty$ → isochoric (= isovolumetric) process

Recall that the internal energy $u = \frac{n}{2}\ kT$, n: degree of freedom

The specific heat capacity $c_v = \left(\frac{\partial u}{\partial T}\right)_v = \frac{n}{2}\ k$,

$$c_p - c_v = k$$

$$\gamma = \frac{c_p}{c_v} = \frac{\frac{n}{2}\ k + k}{\frac{n}{2}\ k} = 1 + \frac{2}{n} = \frac{n+2}{n}$$

For an ideal gas, $n = 3$, so $\gamma = 5/3 \approx 1.66$

For a diatomic gas, $n = 5$, so $\gamma = 7/5 \approx 1.40$

For a photon gas, $n = 6$, so $\gamma = 4/3 \approx 1.33$

$$\frac{dm(r)}{dr} = 4\pi r^2 \rho(\mathrm{r})$$

$$(1)\ \frac{dP}{dr} = -\frac{G\,m(r)}{r^2}\rho \longrightarrow m(r) = -\frac{r^2}{G\rho}\frac{dP}{dr}$$

Plug into (2)

$$\frac{d}{dr}\left(-\frac{r^2}{G\rho}\frac{dP}{dr}\right) = 4\pi r^2 \rho$$

Rearrange to yield

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -4\pi G\rho \ldots (3)$$

Cf. the general Laplace eq. and Poisson eq.

Poisson equation

$\nabla^2\varphi = f$ (if $f = 0 \rightarrow$ Laplace eq.)

$$\text{or } \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\varphi(x, y, z) = f(x, y, z)$$

(1) Gravity

$\nabla \cdot \vec{g} = -4\pi G\rho$, but $\vec{g} = -\nabla\varphi \implies \nabla^2\varphi = 4\pi G\rho$

Solution $\varphi(r) = -\frac{GM}{r}$

(2) Electrostatics

Gauss's law, $\nabla \cdot \vec{D} = \rho_{\text{free}}, \vec{D} = \epsilon\vec{E}, \vec{E} = -\nabla\varphi, \nabla^2\varphi = -\rho/\epsilon$

Solution $\varphi(r) = -\frac{Q}{4\pi\varepsilon r}$

Assume a polytrope; i.e., a spherical fluid with P and ρ being related by

$$P \equiv K\rho^{1+\frac{1}{n}} = K(\rho_c\theta^n)^{1+\frac{1}{n}}$$

$\rho = \rho_c\theta^n$

θ is dimensionless and specifies how density varies with mass

Then (3) becomes $\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -4\pi G\rho$

$$\frac{1}{r^2}\frac{d}{dr}\left[\frac{r^2}{\rho_c\theta^n} K\,\rho_c^{1+\frac{1}{n}}(n+1)\theta^n\,\frac{d\theta}{dr}\right] = -4\pi G\,\rho_c\theta^n$$

And after rearranging

$$\underbrace{\left[\frac{n+1}{4\pi G}K\,\rho_c^{\frac{1}{n}-1}\right]}_{\alpha^2}\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\theta}{dr}\right) = -\theta^n$$

Letting $r = \alpha\xi$, we get

ξ is dimensionless and specifies how radius varies with mass.

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^n$$

$n \nearrow, \rho_c/\bar{\rho} \nearrow$

This is the **Lane-Emden equation** of index n, after J. H. Lane and R. Emden.

Compared to (3), a given n
→ a solution with different K, and ρ_0
→ a family of solutions

The structure of a polytrope depends on n.

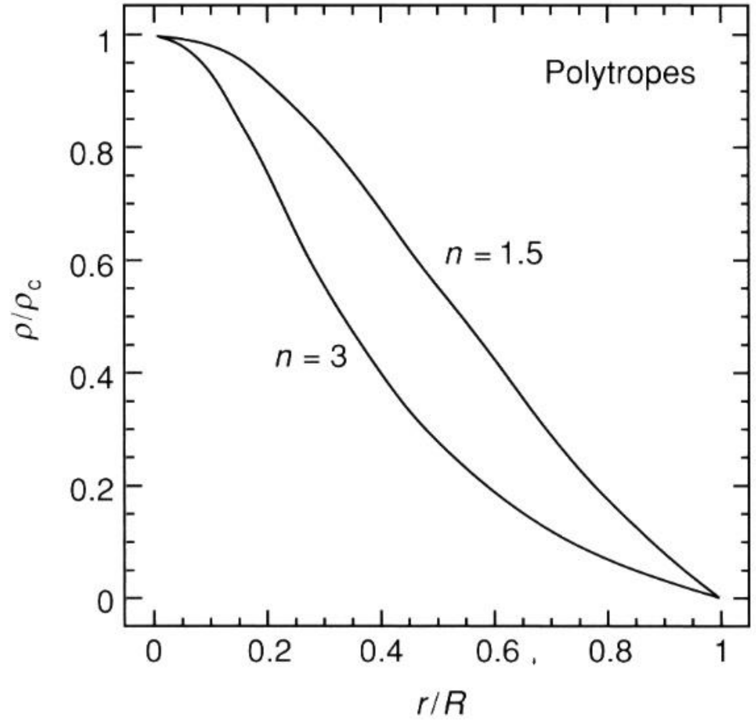


Figure 5.1 Normalized polytropes for $n = 1.5$ and $n = 3$.

The Lane-Emden equation has the boundary conditions of $\theta = 1$ and $\frac{d\theta}{d\xi} = 0$ at $\xi = 0$, and can be integrated from $\xi = 0$.

For $n = 0, 1, 5$, analytic solutions are available; otherwise the integration is done numerically.

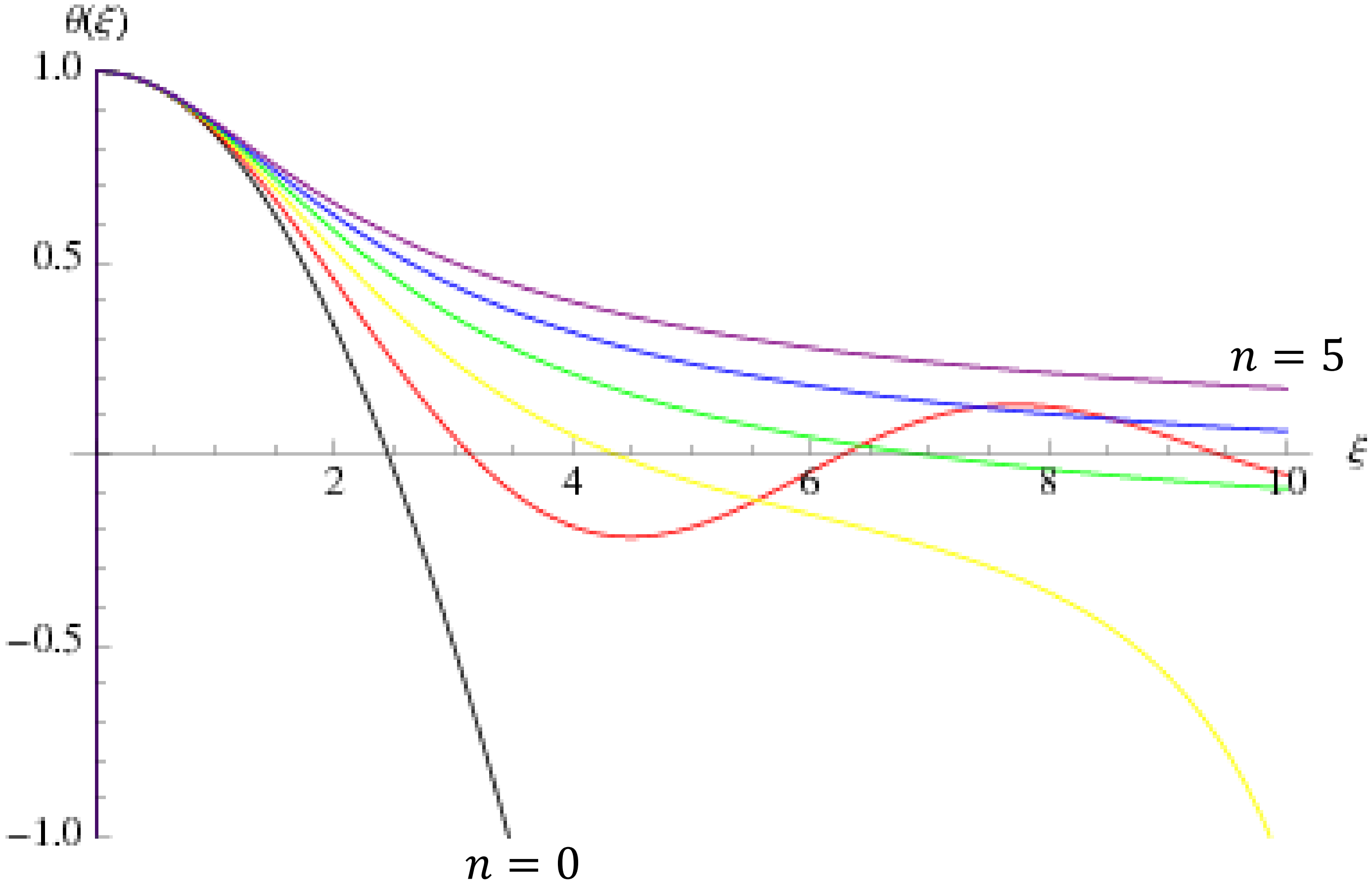
$n = 0$, $\theta_0 = 1 - \xi^2/6$ $\qquad$ $\theta_0 = 1 - \xi^2/6 = 0 \Longrightarrow \xi_1 = \sqrt{6}$

$n = 1$, $\theta_1 = \sin\xi/\xi$ $\qquad$ $\theta_1 = \sin\xi/\xi = 0 \Longrightarrow \xi_1 = \pi$

$n = 5$, $\theta_5 = (1 + \xi^2/3)^{-1/2}$ $\qquad$ $\theta_5 = (1 + \xi^2/3)^{-1/2} \Longrightarrow \xi_1 = \infty$

For $n = 0$ and $n = 1$, solution $\rightarrow 0$ at some point ($\rho \rightarrow 0$); this defines the boundary of the star, i.e., ξ at first zero (ξ_1)=radius. Solve $\theta_{\mathrm{n}}(\xi_1) = 0$.

For $n = 0$, $\rho = \rho_{\mathrm{c}}\theta^0$ =const; for $n = 5$, solution never goes to 0.



$n = 0$, a constant density sphere; $\xi_1 = \sqrt{6}; P = P_c\theta$

$n = 1$, solution a sync function; $\xi_1 = \pi;\ \rho = \rho_c\theta; P = P_c\theta^2$

$n = 5$, finite density, but infinite radius; $\xi_1 \to \infty$

Weisstein, Eric W. "Lane-Emden Differential Equation." From *MathWorld*--A Wolfram Web Resource. http://mathworld.wolfram.com/Lane-EmdenDifferentialEquation.html

The Lane-Emden equation is integrated often numerically to the first zero. The overall stellar properties can then be computed.

Mass

$$M(\xi) = \int_0^{\alpha\xi} 4\pi\rho r^2 dr = 4\,\pi\alpha^3\rho_c\left[-\xi^2\,\frac{d\theta}{d\xi}\right]_{\xi=\xi_1}$$

Radius

$$R = \alpha\xi_1$$

Central pressure

$$P_c = \frac{GM^2}{R^4}\left[4\,\pi(n+1)\,\left(\frac{d\theta}{d\xi}\right)^2_{\xi=\xi_1}\right]^{-1}$$

Mean density

$$\bar{\rho} = \rho_c\left[-\frac{3}{\xi}\,\frac{d\theta}{d\xi}\right]_{\xi=\xi_1}$$

Bower & Deeming

Gravitational binding energy

$$\Omega = -\frac{3}{5-n}\frac{GM^2}{R}$$

For $n = 5, \Omega \to -\infty$. For any $n > 5$ (i.e., $\gamma < 6/5$), $\Omega > 0$, the system is not gravitationally bound; no stable configuration

Given a solution $\theta(\xi)$, i.e., $\rho(r)$, the density and pressure profiles can be derived.

mass | density | P_c

TABLE 4

THE CONSTANTS OF THE LANE-EMDEN FUNCTIONS*

n	ξ_1	$-\xi_1^2\left(\frac{d\theta_n}{d\xi}\right)_{\xi=\xi_1}$	$\rho_c/\bar{\rho}$	$\omega_n = -\xi_1^{\frac{n+1}{n-1}}\left(\frac{d\theta_n}{d\xi}\right)_{\xi=\xi_1}$	N_n	W_n	$-\frac{1}{(n+1)\xi_1\left(\frac{d\theta_n}{d\xi}\right)_{\xi=\xi_1}}$
0..........	2.4494	4.8988	1.0000	0.33333	....	0.119366	-0.5
0.5..........	2.7528	3.7871	1.8361	0.02156	2.270	0.26227	0.53847
1.0..........	3.14159	3.14159	3.28987	....	0.63662	0.392699	0.5
1.5..........	3.65375	2.71406	5.99071	132.3843	0.42422	0.770140	0.53849
2.0..........	4.35287	2.41105	11.40254	10.4950	0.36475	1.63818	0.60180
2.5..........	5.35528	2.18720	23.40646	3.82662	0.35150	3.90906	0.69956
3.0..........	6.89685	2.01824	54.1825	2.01824	0.36394	11.05066	0.85432
3.25..........	8.01894	1.94980	88.153	1.54716	0.37898	20.365	0.96769
3.5..........	9.53581	1.89056	152.884	1.20426	0.40104	40.9098	1.12087
4.0..........	14.97155	1.79723	622.408	0.729202	0.47720	247.558	1.66606
4.5..........	31.83646	1.73780	6189.47	0.394356	0.65798	4922.125	3.33100
4.9..........	169.47	1.7355	934800	0.14239	1.340	3.693×10^6	16.550
5.0..........	∞	1.73205	∞	0	∞	∞	∞

* The values for $n = 0.5$ and 4.9 are computed from Emden's integrations of θ_n; for $n = 3.25$ an unpublished integration by Chandrasekhar has been used. $n = 5$ corresponds to the Schuster-Emden integral. For the other values of n the *British Association Tables*, Vol. II, has been used.

$$N_n = \frac{(4\pi)^{1/n}}{n+1}\left[-\xi_1^{\,n+1/n-1}\left(\frac{d\theta}{d\xi}\right)_{\xi=\xi_1}\right]^{1-n}$$

$$W_n = \frac{1}{4\pi(n+1)\left[\left(\frac{d\theta}{d\xi}\right)_{\xi=\xi_1}\right]^{-2}}$$

Chandrasekhar p.96

Table 2-5 Constants of the Lane-Emden functions†

n	ξ_1	$-\xi_1^2 \left(\frac{d\phi}{d\xi}\right)_{\xi=\xi_1}$	$\frac{\rho_c}{\bar{\rho}}$
0	2.4494	4.8988	1.0000
0.5	2.7528	3.7871	1.8361
1.0	3.14159	3.14159	3.28987
1.5	3.65375	2.71406	5.99071
2.0	4.35287	2.41105	11.40254
2.5	5.35528	2.18720	23.40646
3.0	6.89685	2.01824	54.1825
3.25	8.01894	1.94980	88.153
3.5	9.53581	1.89056	152.884
4.0	14.97155	1.79723	622.408
4.5	31.83646	1.73780	6,189.47
4.9	169.47	1.7355	934,800
5.0	∞	1.73205	∞

† S. Chandrasekhar, "An Introduction to the Study of Stellar Structure," p. 96; reprinted from the Dover Publications edition, Copyright 1939 by The University of Chicago, as reprinted by permission of The University of Chicago.

The case for $n = 0$, $\rho = \rho_c \theta^0 = \text{const.}$

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2 \frac{d\theta}{d\xi}\right) = -\theta^n$$

$$\frac{d}{d\xi}\left(\xi^2 \frac{d\theta}{d\xi}\right) = -\xi^2$$

$$\xi^2 \frac{d\theta}{d\xi} = -\frac{1}{3}\xi^3 + c_1$$

$$\frac{d\theta}{d\xi} = -\frac{1}{3}\xi + \frac{c_1}{\xi^2}$$

$$\theta = -\frac{1}{6}\xi^2 - \frac{c_1}{\xi} + c_2$$

For the integration constants, c_1 must be zero to avoid singularity at origin.

Because $\rho = \rho_c$ at $\theta = 1$, $c_2 = 1$

$$\longrightarrow \theta(\xi) = 1 - \frac{1}{6}\xi^2$$

$$\xi_1 = \xi(\theta = 0) = \sqrt{6}$$

Recall $\rho = \rho_c \theta^n$, and $r = \alpha\xi$,

$$M = \int_0^R 4\pi r^2 \rho \, dr = 4\,\pi\alpha^3\rho_c \int_0^{\xi_1} \xi^2 \theta^n \, d\xi$$

(from Lane-Emden eq.)

$$= 4\,\pi\alpha^3\rho_c \int_0^{\xi_1} \left[-\frac{d}{d\xi}\left(\xi^2 \frac{d\theta}{d\xi} \right) \right] d\xi$$

If the star is supported by both radiation pressure and gas pressure, the total pressure $P = P_{\text{gas}} + P_{\text{rad}}$.

Define $\beta = P_{\text{gas}}/P$.

$$P_{rad} = \frac{4\sigma}{3c}T^4 = \frac{1}{3}aT^4 = (1-\beta)P$$

For ideal gas, $P_{\text{gas}} = \frac{\rho}{\mu m_H}kT = \beta P$

Eliminate T, $T = \mu m_H \beta P/\rho k$, into $T^4 = 3(1-\beta)P/a$

$P = K\rho^{4/3} \longrightarrow \gamma = 4/3$ or $n = 3$

This is the Eddington standard model ($n = 3$).

A special case --- an isothermal gas sphere $P \propto \rho$

This is a polytrope of $\gamma = 1$, or $n \rightarrow \infty$

n >5, so the sphere is infinite in extent. Need to work out the solution from beginning.

Recall Eq. 3, $\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -4\pi G\rho$

Plug in the ideal gas equation of state, $P = \rho kT/\mu m_H$

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{kT}{\mu m_H}\frac{d\rho}{dr}\right) = -4\pi G\rho$$

Let $\rho = \rho_c e^{-\psi}, r = \left[\frac{kT}{4\pi G\mu m_H \rho_c}\right]^{1/2} \xi = \alpha\xi.$

The equation becomes

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2 \frac{d\psi}{d\xi}\right) = e^{-\psi}$$

with the BCs, $\psi = 0$, $\mathrm{d}\psi/\mathrm{d}\xi = 0$, and $\xi = 0$.

This must be solved numerically, and the solution diverges (i.e., density never goes to 0, and mass goes to infinitive.)

Conclusion: A finite star cannot be an isothermal gas sphere.

Star Formation in a Nutshell

- Stars are formed in groups out of dense molecular cloud cores. Planets are formed in young circumstellar disks.
(*Jeans criteria*)

- Initial gravitational contraction leads to a decrease of luminosity, while surface temperature remains almost unchanged.
(*Pre-main sequence Hayashi track*)

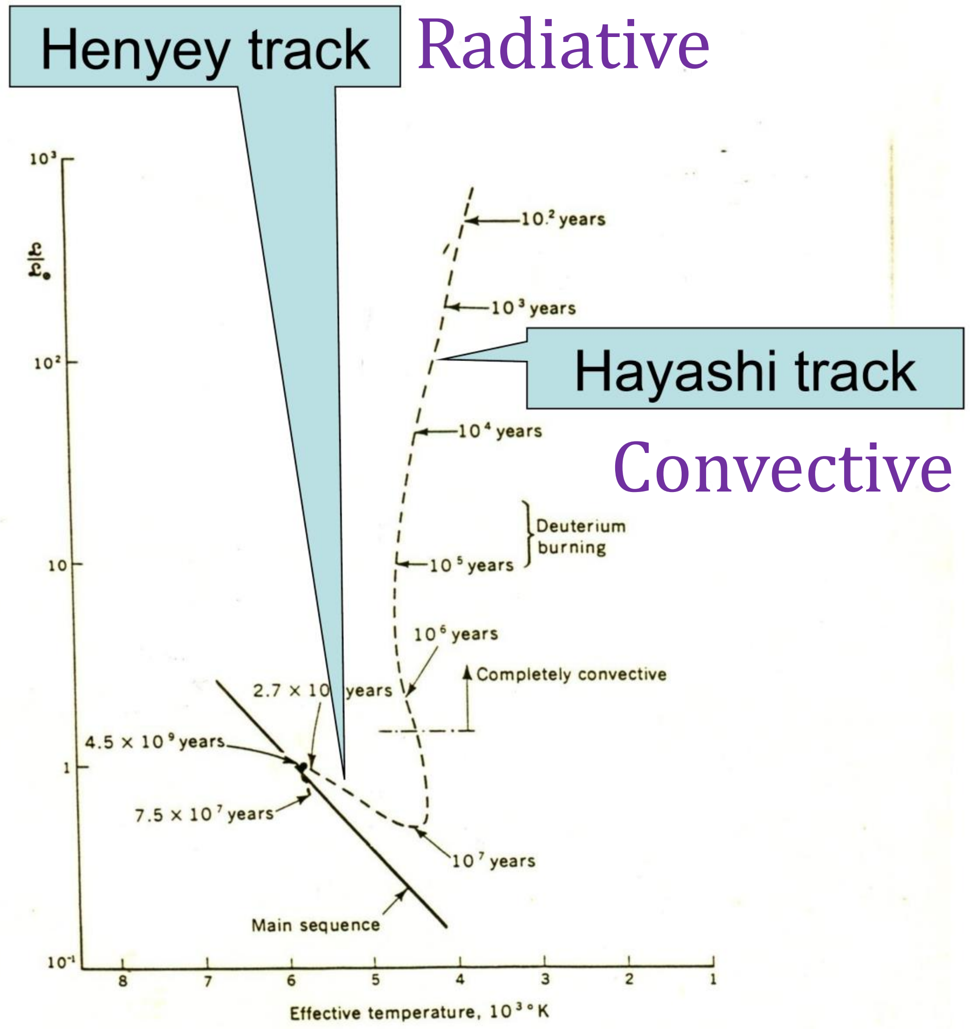


Fig. 5-1 The path on the H-R diagram of the contraction of the sun to the main sequence. The interior has become sufficiently hot to burn deuterium after about 10^5 years. The contraction ceases near the main sequence when the core has become hot enough to replenish the solar luminosity with the thermonuclear power generated by the fusion of hydrogen into helium. [*After D. Ezer and A. G. W. Cameron, The Contraction Phase of Stellar Evolution, in R. F. Stein and A. G. W. Cameron (eds.), "Stellar Evolution," Plenum Press, New York, 1966.*]

1965 ApJ, 141, 993

STELLAR EVOLUTION. I. THE APPROACH TO THE MAIN SEQUENCE*

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ABSTRACT

The manner in which nuclear reactions replace gravitational contraction as the major source of stellar luminosity is investigated for model stars of population I composition in the mass range $0.5 < M/M_{\odot} <$ 15 0. By following in detail the depletion of C^{12} from high initial values down to values corresponding to equilibrium with N^{14} in the C-N cycle, the approach to the main sequence in the Hertzsprung-Russell diagram and the time to reach the main sequence, for stars with $M \geq 1.25\, M_{\odot}$, are found to differ significantly from data reported previously.

Zero-age main sequence (ZAMS): the locus in the HRD of stars of different masses first reaching the main sequence (i.e., starting steady core H fusion)

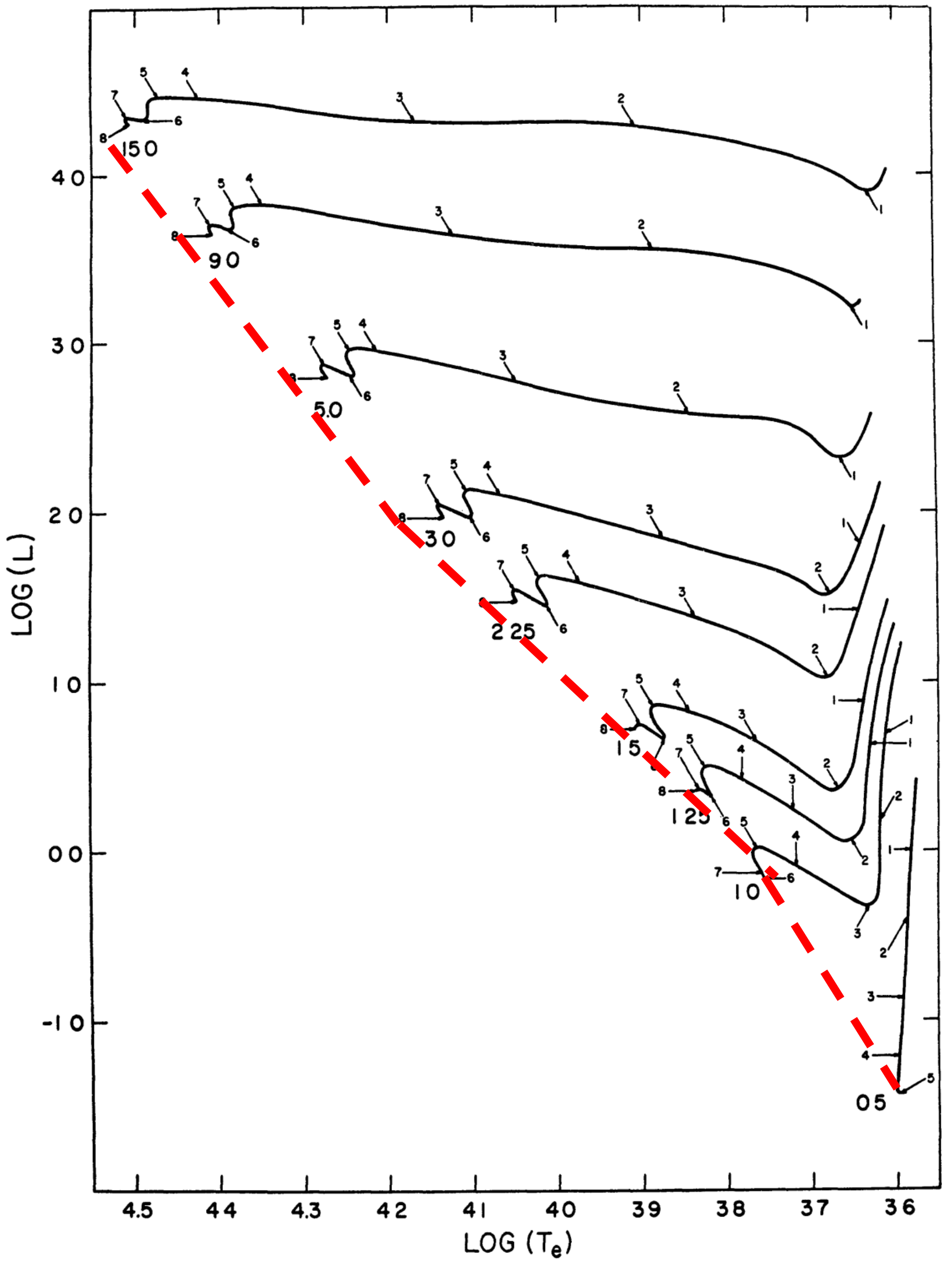
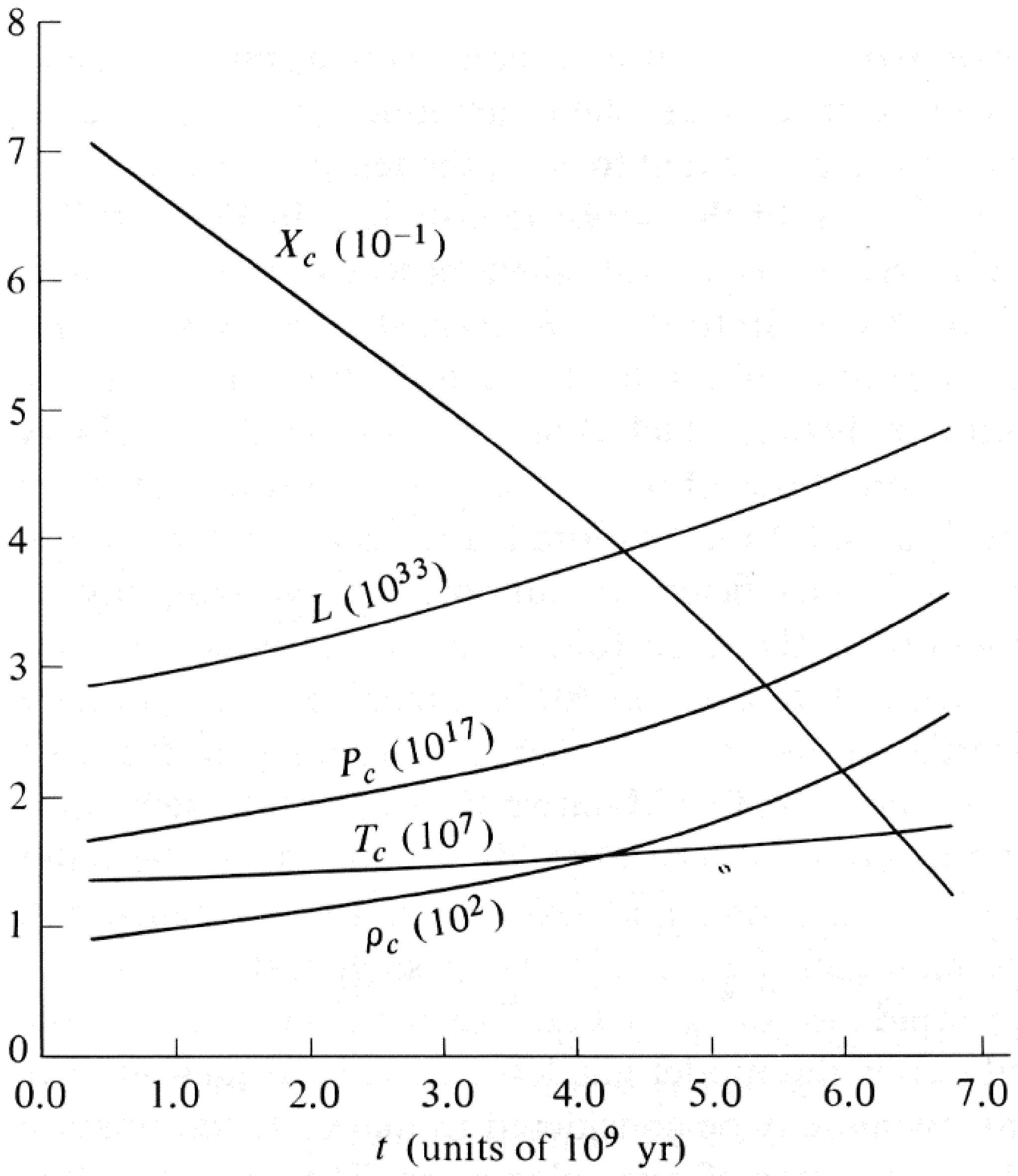


FIG. 17.—Paths in the Hertzsprung-Russell diagram for models of mass $(M/M_{\odot})$ = 0.5, 1.0, 1.25, 1.5, 2.25, 3 0, 5.0, 9.0, and 15.0. Units of luminosity and surface temperature are the same as those in Fig. 1

P, ρ and L in [cgs]

T in [K]



The evolution of the Sun, from

$X = 0.730,$

$Y = 0.245,$

$Z = 0.025$

Novotny

Thermonuclear Reactions

- Eddington in 1920s hypothesized that fusion reactions between light elements were the energy source of stars.

- Stellar evolution = (con) sequences of nuclear reactions

- $E_{\text{kinetic}} \approx kT_c \approx 8.62 \times 10^{-8}\, T \sim$ keV,

  but $E_{\text{Coulomb barrier}} = \frac{Z_1 Z_2 e^2}{r} = \frac{1.44\, Z_1 Z_2}{r[\text{fm}]} \sim$ MeV.

  This is 3 orders higher than the kinetic energy of the particles.

- Tunneling effect in QM proposed by Gamow (1928, Z. Physik, **52**, 510);
  applied to energy source in stars by Atkinson
  & Houtermans (1929, Z. Physik, **54**, 656)

Quantum mechanics tunneling effect

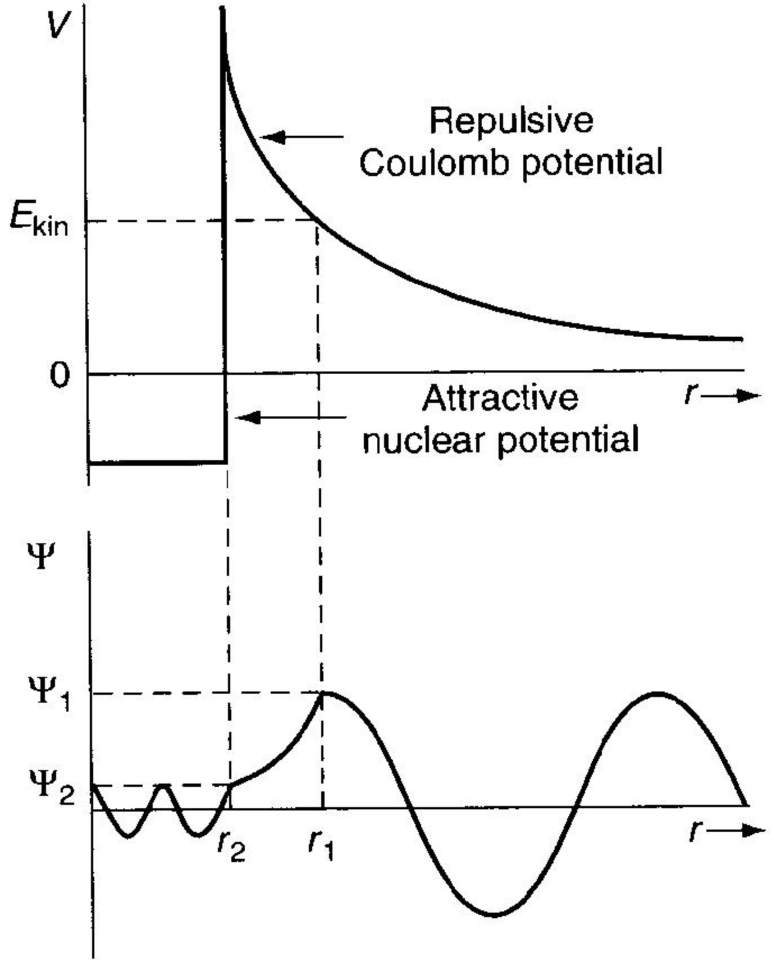


Figure 3.4 Illustration of the potential seen by particle b when approaching particle A with a kinetic energy E_{kin}, and the corresponding wavefunction Ψ; classically, particle b would reach only a distance r_1 from particle A before being repelled by the Coulomb force

Cross section for nuclear reactions (penetrating probability)

$$\propto e^{-\pi Z_1 Z_2 e^2/\varepsilon_0 h v}$$

This ↗ as v ↗

Velocity probability distribution (Maxwellian)

$$\propto e^{-mv^2/2kT}$$

This ↘ as v ↗

∴ Product of these 2 factors ➔ Gamow peak

D. Clayton "Principles of Stellar Evolution and Nucleosynthesis"

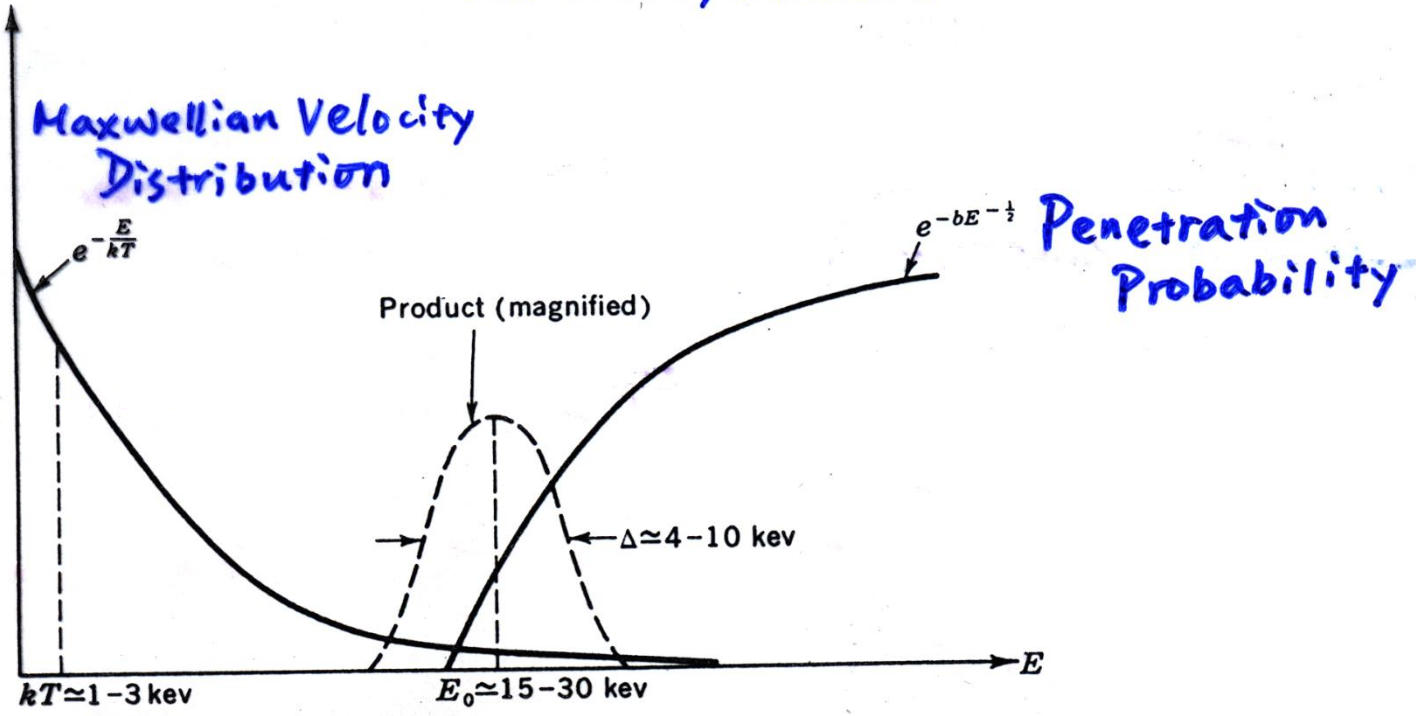


Fig. 4-6 The dominant energy-dependent factors in thermonuclear reactions. Most of the reactions occur in the high-energy tail of the maxwellian energy distribution, which introduces the rapidly falling factor $\exp(-E/kT)$. Penetration through the coulomb barrier introduces the factor $\exp(-bE^{-\frac{1}{2}})$, which vanishes strongly at low energy. Their product is a fairly sharp peak near an energy designated by E_0, which is generally much larger than kT. The peak is pushed out to this energy by the penetration factor, and it is therefore commonly called the *Gamow peak* in honor of the physicist who first studied the penetration through the coulomb barrier.

Clayton

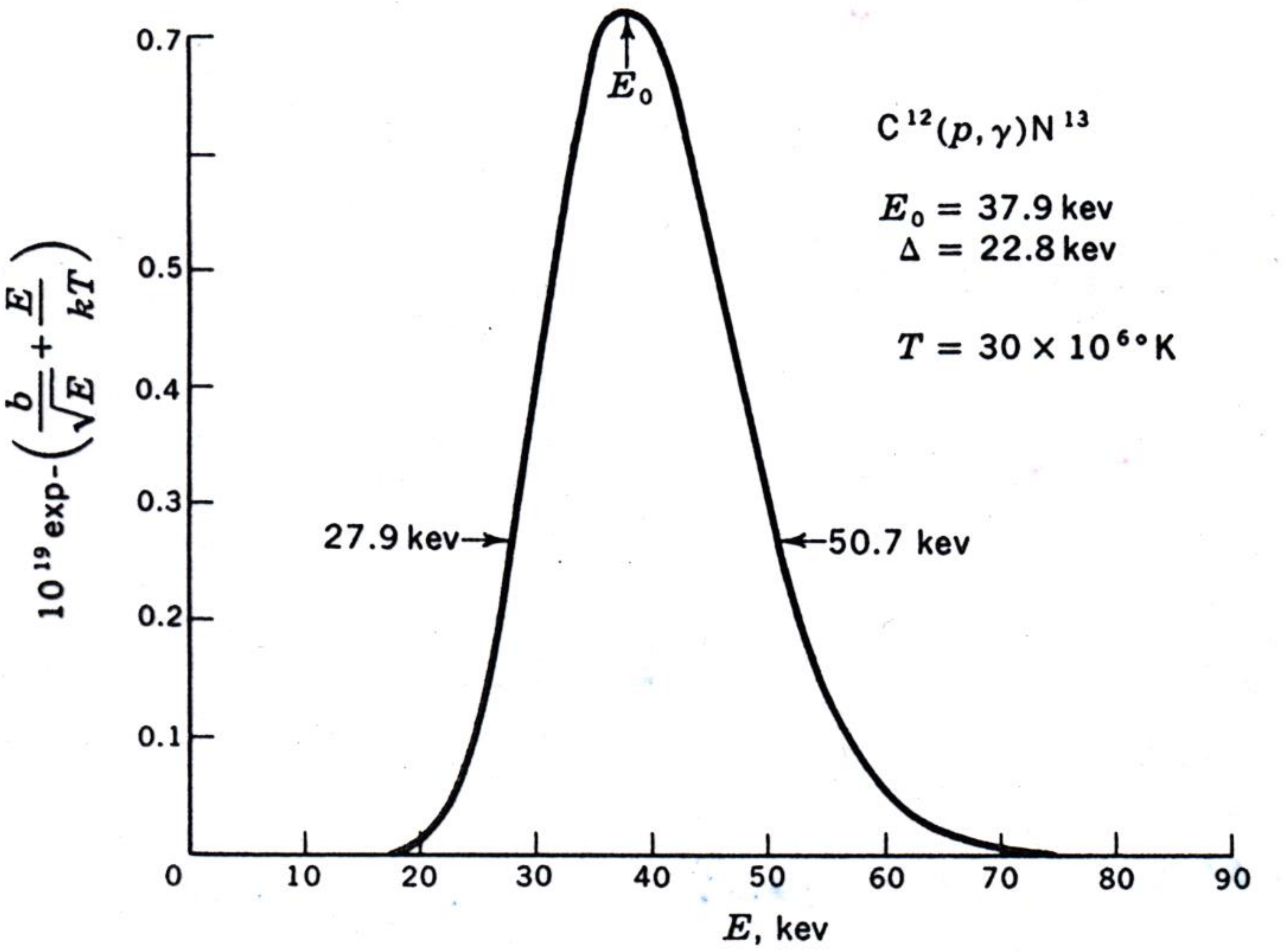


Fig. 4-7 The Gamow peak for the reaction $C^{12}(p,\gamma)N^{13}$ at $T = 30 \times 10^6$ °K. The curve is actually somewhat asymmetric about E_0, but it is nonetheless adequately approximated by a gaussian.

Clayton

Resonance → very sharp peak in the reaction rate

So there exists a narrow range of temperature in which the reaction rate ↑↑
→ a power law

→ an "ignition" (threshold) temperature

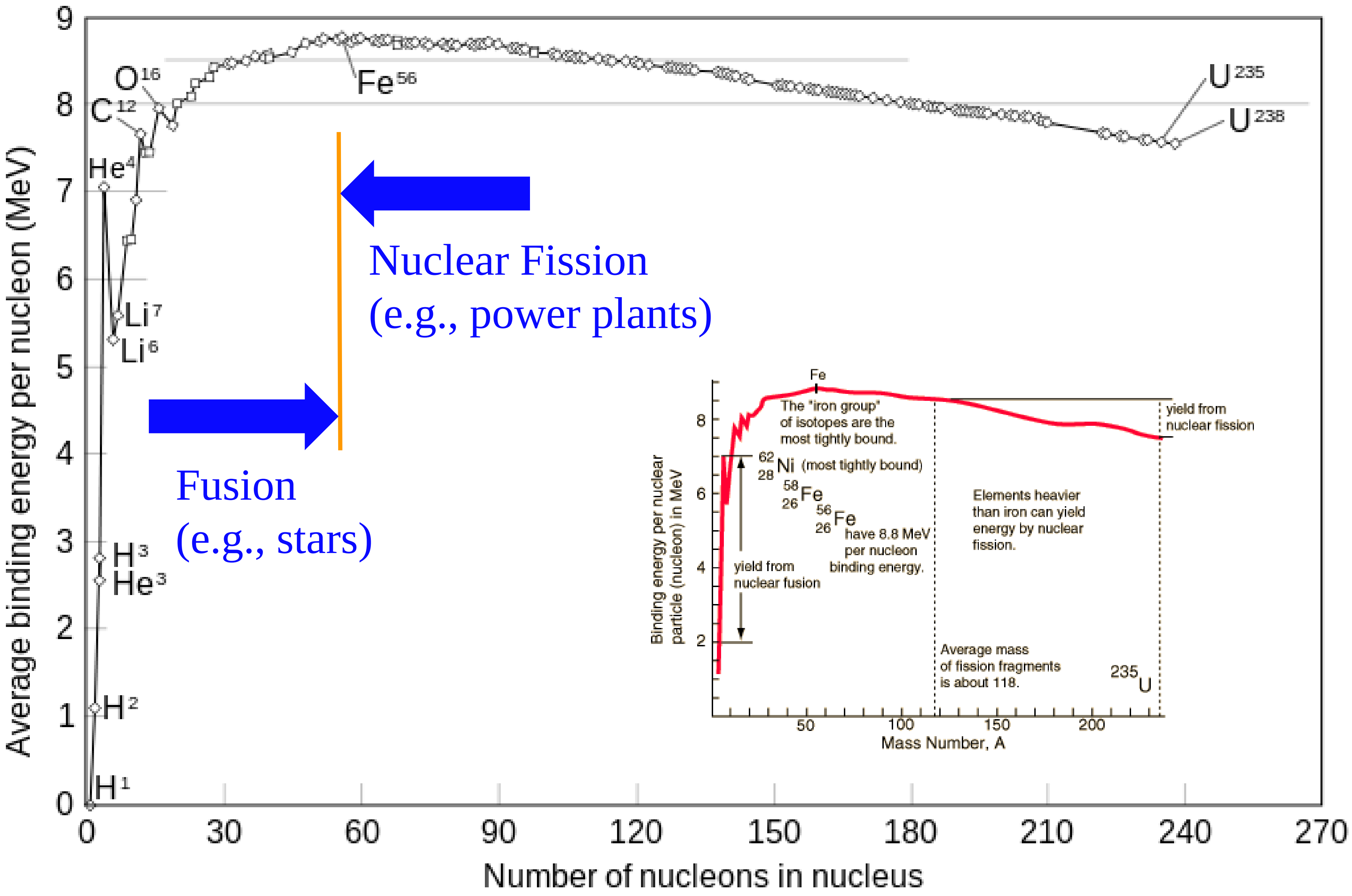
Resonance reactions
Energy of interacting particles ≈ Energy level of compound nucleus

For a thermonuclear reaction or a nucleosynthesis (fusion) process, the reaction rate is expressed as

$$q \propto \rho^m T^n$$

Nuclear reaction rate

- ✓ $r_{12} \propto n_1 n_2 \langle \sigma v \rangle \propto n_1 n_2 \exp\left[-C\left(\frac{z_1^2 z_2^2}{T_6}\right)^{1/3}\right]$ [cm^{-3}s^{-1}]
- ✓ As $T \nearrow$, $r_{12} \nearrow\nearrow$
- ✓ Major reactions are those with smallest $Z_1 Z_2$, i.e., lowest Coulomb barriers.
- ✓ n_i is the particle volume number density, $n_i m_i = \rho X_i$, where X_i is the mass fraction
- ✓ $q_{12} \propto Q\, \rho\, X_1\, X_2 / m_1 m_2$ [erg g^{-1}s^{-1}]


Average binding energy per nucleon (MeV)
Number of nucleons in nucleus
0
1
2
3
4
5
6
7
8
9
0
30
60
90
120
150
180
210
240
270
H1
H2
H3
He3
He4
Li6
Li7
C12
O16
Fe56
U235
U238
Nuclear Fission
(e.g., power plants)
Fusion
(e.g., stars)
Fe
The "iron group" of isotopes are the most tightly bound.
62 28 Ni (most tightly bound)
58 26 Fe
56 26 Fe
have 8.8 MeV per nucleon binding energy.
yield from nuclear fusion
Elements heavier than iron can yield energy by nuclear fission.
yield from nuclear fission
Average mass of fission fragments is about 118.
235 U
Binding energy per nuclear particle (nucleon) in MeV
Mass Number, A
50
100
150
200

Binding Energy per Nucleon

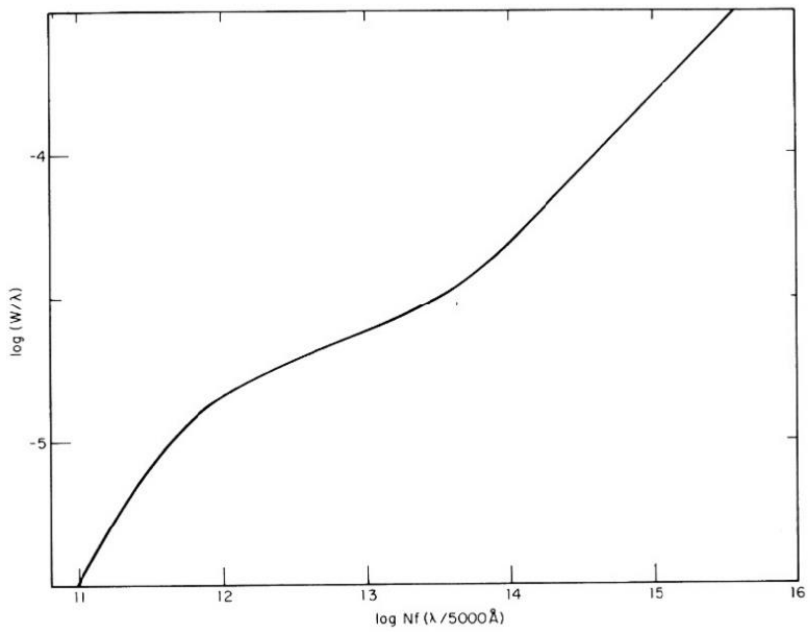
Z	A	Symbol	B (MeV)/A
0	1	n	0.0
1	1	H	0.0
	2	D	1.112260
	3	T	2.827307
2	3	He	2.572693
	4		7.074027
3	6	Li	5.332148
	7		5.606490

Arnett

Stellar Atmosphere and Structure

Problem Set #20201203, due in two weeks

1. With the attached figure and table, compute the number of neutral sodium atoms in the ground state on the sun's surface.



From Aller

Data for solar sodium lines

λ	W (mA)	f
3302.38	88	0.0214
3302.98	67	0.0049
5889.97	730	0.645
5895.94	560	0.325

2. A star of mass M and a homogeneous composition assumes a density of a radial dependence, $\rho(r) = \rho_0 \, [1 - (r/R_0)^2]$, where ρ_0 is the central density, and R_0 is the radius of the star. (a) Find $m(r)$. (b) Find the relation between M and R_0. (c) Derive and plot the pressure as a function of radius. (d) What is the central temperature of the star?

3. The Lane-Emden equation for stellar structure is a form of Poisson equation,

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2 \frac{d\theta}{d\xi}\right) = -\theta^n,$$

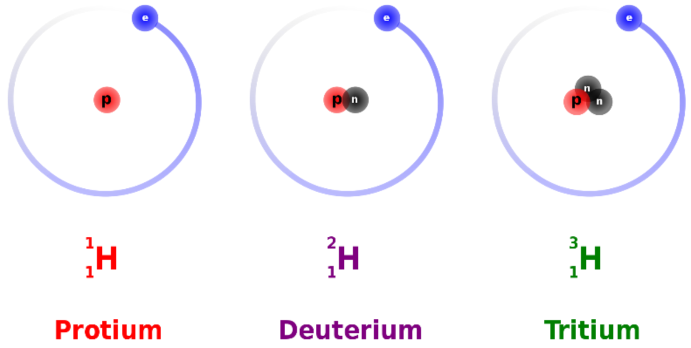
where ξ is a dimensionless radius, θ, also dimensionless, describes the density profile of the star, and n is the polytropic index. (a) Derive the equation, and describe what each symbol stands for. (b) Solve it analytically for $n = 1$. Find the total mass of the star $M = M(R)$, where R is the stellar radius. (c) Assume $n = 3$ for the Sun, compute the central density, central pressure, and central temperature. Compare the computed central temperature with the currently best estimated central temperature.

Deuterium Burning

$M_\odot c^2 = 2 \times 10^{54}$ ergs
1 amu = 931 Mev/c^2

$$^2H + {}^1H \longrightarrow {}^3He + \gamma \quad (T > 10^6 K)$$

$$^2H({}^1H, \gamma){}^3He$$



Deuterium: D or ^{2}H, with the nucleus consisting of 1 p$^+$ and 1 n^0

$$Q_{DP} = 5.5 \text{ MeV}$$

$$q_{DP} = 4.19 \times 10^7 [D/H] \left(\frac{\rho}{1 \text{ g cm}^{-3}}\right)\left(\frac{T}{10^6 K}\right)^{11.8} [\text{erg g}^{-1} \text{s}^{-1}]$$

ISM value, $\langle D/H \rangle \sim 2 \times 10^{-5}$

Earth ocean 1.6×10^{-4}

D/H

- 156 ppm … Terrestrial seawater (1.56×10^{-4})
- 22~26 ppm … Jupiter
- 17 ppm … Saturn
- 55 ppm … Uranus
- 200 ppm … Halley's Comet

$n + p \rightarrow D + \gamma$ (production of D)

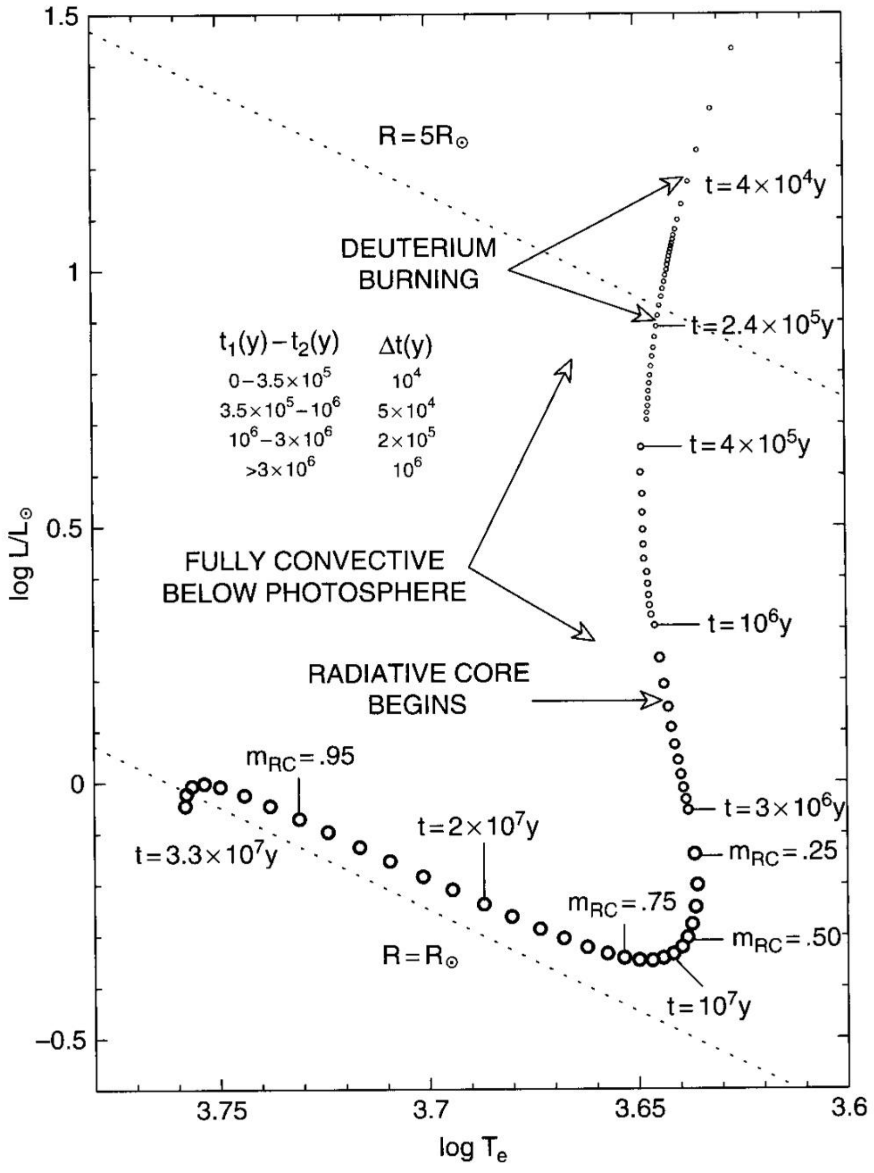
$D + D \rightarrow {}^{4}He + \gamma$ (destruction) → faster

The lower the mass density, the more the D abundant
→ D as a sensitive tracer of the density of the early Universe

Before the Big Bang nucleosynthesis, there were plenty of neutrons, but much less abundant than protons, so all neutrons go into making ${}^{4}He$

$$\rightarrow {}^{4}He \approx \frac{n/2}{(n+p)/4} = \frac{2n}{n+p}$$

Current value $n/p \approx 0.12$, so ${}^{4}He \approx 2/9$, as observed today.



Pre main sequence evolution in the HR diagram of a low mass model ($M = 1M_\odot, Z = 0.01, Y = 0.25$)

^{2}D burns at $T \approx 10^6$ K

$$^2H(^1H,\gamma)^3He$$

^{7}Li burns at $T \approx 3 \times 10^6$ K

$$^7Li(^1H,\gamma)^4He$$

^{1}H burns at $T \approx 5 \times 10^6$ K

Iben 2013

Hydrostatic equilibrium

$$\frac{dP}{dr} = -\frac{Gm(r)}{r^2}\rho, \text{ so } \frac{P}{R} = \frac{GM}{R^2}\frac{M}{R^3} \rightarrow \quad \boxed{P = \frac{GM^2}{R^4}}$$

Force/Area

Ideal gas law

$$P = \frac{\rho}{\mu m_H} kT; \ \rho = \frac{M}{R^3} \rightarrow \boxed{P = \frac{MT}{R^3\mu}\frac{k}{m_H}}$$

Equating the two pressure terms → $T \sim \frac{\mu\, GM}{R}$

This should be valid at the star's center, thus

$$\boxed{T_* \sim \frac{\mu\, GM_*}{R_*}}$$

Recall a star's central temperature

$$T_c \sim \frac{\mu G M}{R} \cdot \alpha$$

mass distr.

Numerically

$$T_c = 7.5 \times 10^6\,K \left(\frac{M_*}{M_\odot}\right)\left(\frac{R_*}{R_\odot}\right)^{-1}$$

$$\therefore \quad M_* = 0.4\,M_\odot \longrightarrow T_c \sim 10^6\,K$$

Lithium Burning

$^7Li + {}^1H \longrightarrow {}^4He + {}^4He$ ($T > 3 \times 10^6 K$)

$Li^7(p, \alpha)He^4$
$Li^6(p, \alpha)He^3$

ISM $[Li/H] \sim 2 \times 10^{-9}$

Primordial abundance 10 x lower,
produced by cosmic rays α hitting ^4He
(inverse reaction)

Li measurable in stellar spectra

Li I 6708Å absorption

(actually doublet 6707.78 and 6707.93
but difficult to resolve

Low-mass protostars, T_c too low to ignite Li fusion, so inherit the full ISM Li supply.

Higher-mass protostars can burn and destroy Li promptly, but the base of the convection zone is below 3×10^6 K, so the surface lithium abundance = ISM value.

Stahler & Palla

Presence of Li I λ6707 absorption → stellar youth

Ca I λ6718 prominent in late-type stars

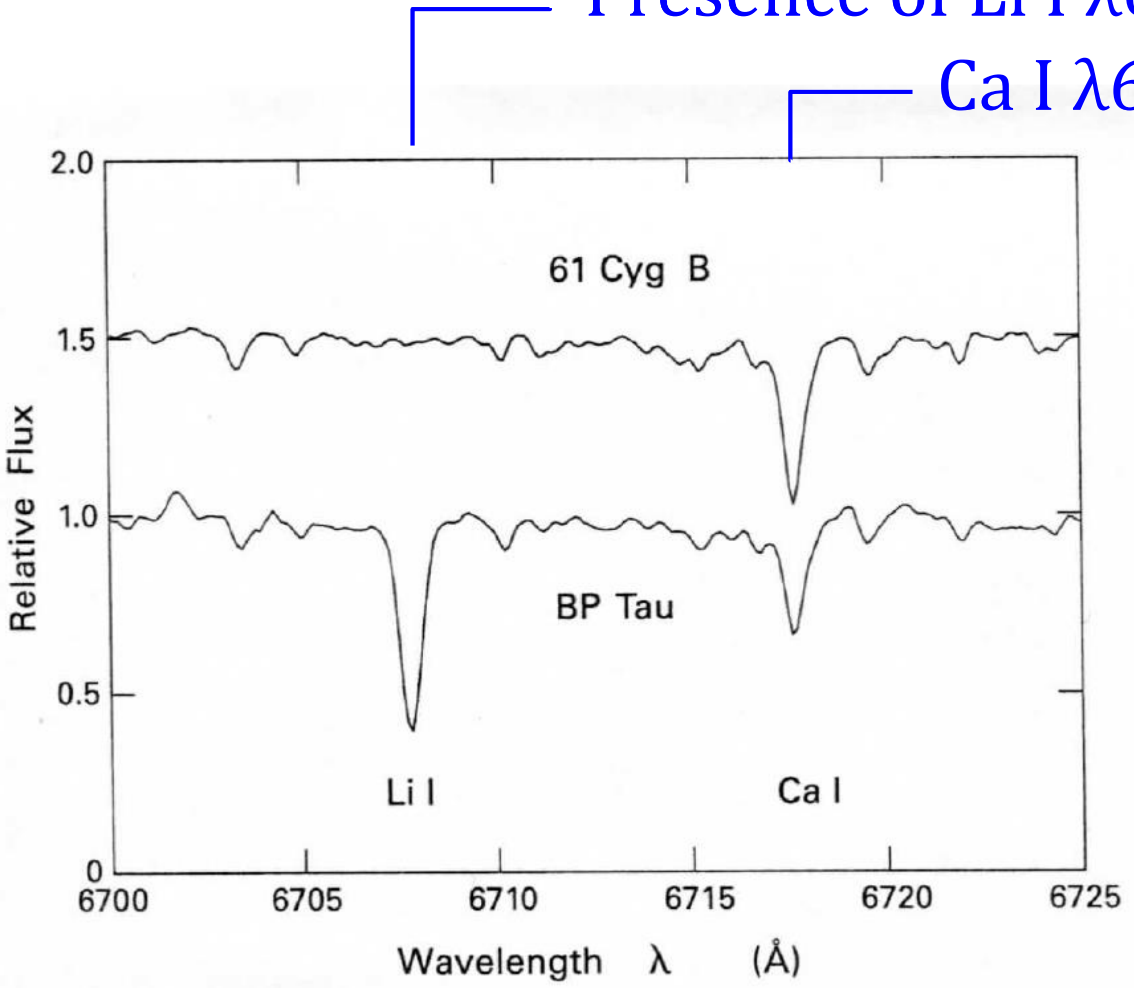


Figure 16.9 Lithium absorption in a pre-main-sequence star. Shown is a portion of the optical spectrum of BP Tau, a T Tauri star of spectral type K7, corresponding to an effective temperature of 4000 K. Also shown, for comparison, is a main-sequence star of the same spectral type, 61 Cyg B. Only in the first star do we see the Li I absorption line at 6708 Å. Both objects also have a strong line due to neutral calcium.

Stahler & Palla

Stars	$\mathcal{M}/\mathrm{M}_\odot > 0.08$, core H fusion Spectral types O, B, A, F, G, K, M
Brown Dwarfs	$0.065 > \mathcal{M}/\mathrm{M}_\odot > 0.013$, core D fusion $0.080 > \mathcal{M}/\mathrm{M}_\odot > 0.065$, core Li fusion Spectral types M6.5–9, L, T, Y Electron degenerate core ✓$10\ \mathrm{g\ cm^{-3}} < \rho_c < 10^3\ \mathrm{g\ cm^{-3}}$ ✓$T_c < 3 \times 10^6$ K
Planets	$\mathcal{M}/\mathrm{M}_\odot < 0.013$, no fusion ever

$1\ M_\odot \approx 1000\ M_J$

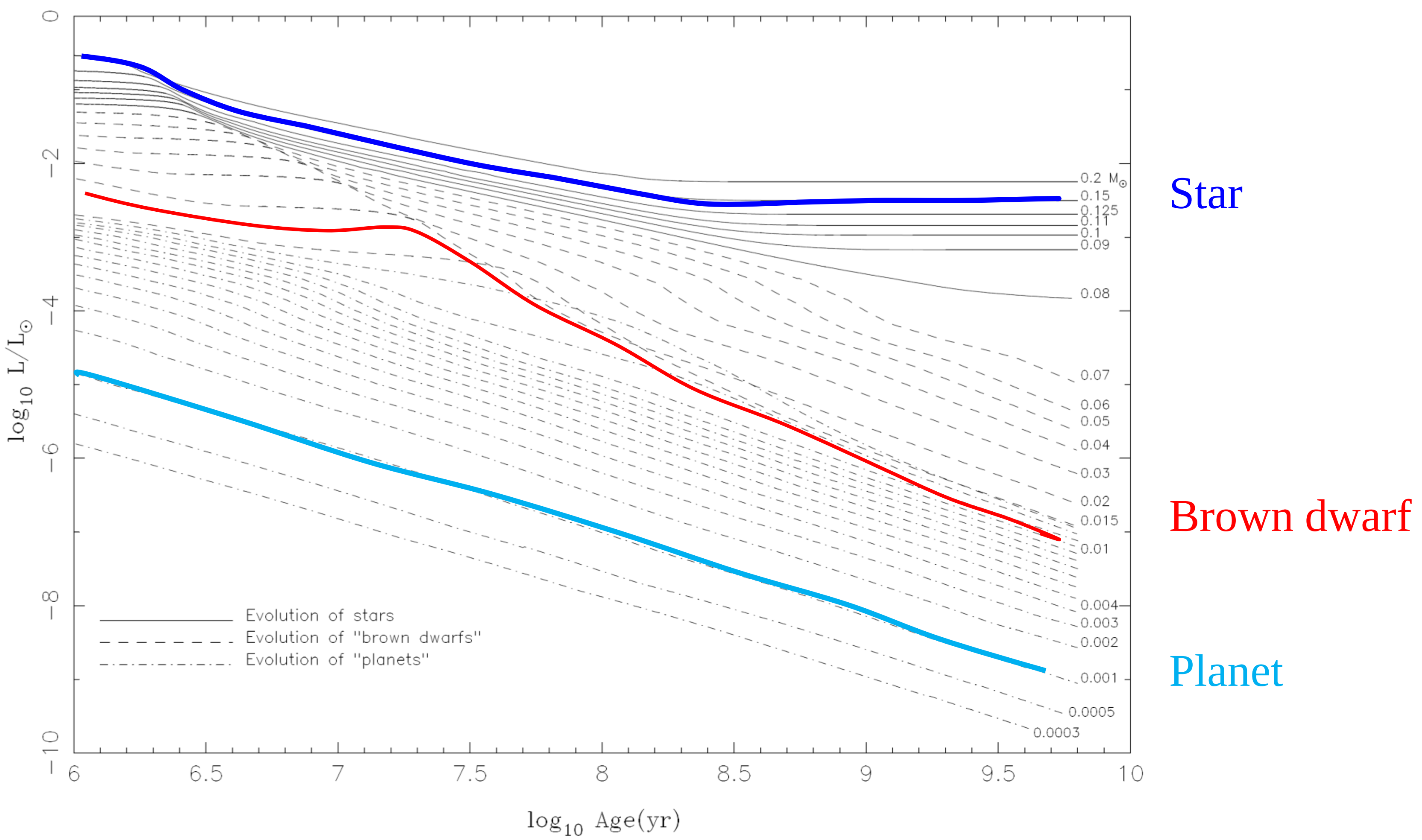
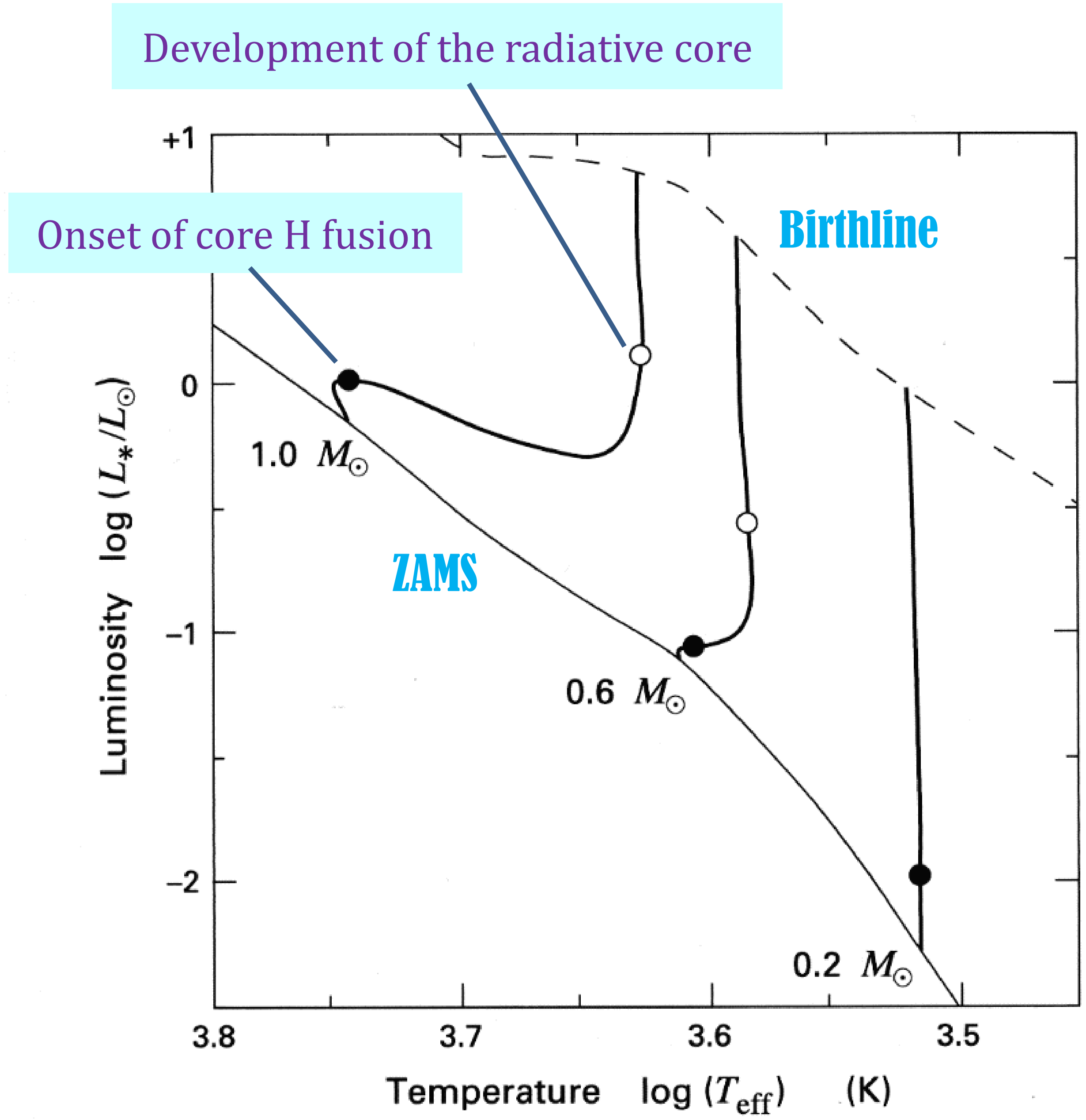


FIG. 7.—Evolution of the luminosity (in $L_\odot$) of solar-metallicity M dwarfs and substellar objects vs. time (in yr) after formation. The stars, "brown dwarfs" and "planets" are shown as solid, dashed, and dot-dashed curves, respectively. In this figure, we arbitrarily designate as "brown dwarfs" those objects that burn deuterium, while we designate those that do not as "planets." The masses (in $M_\odot$) label most of the curves, with the lowest three corresponding to the mass of Saturn, half the mass of Jupiter, and the mass of Jupiter.

Burrows



Stahler & Palla, Fig 16.8

The proton-proton chain

This neutrino carries away 0.26 MeV

$^{1}\mathrm{H} + {}^{1}\mathrm{H} \rightarrow {}^{2}\mathrm{D} + e^{+} + \nu_e$ (1.44 MeV, 1.4×10^{10} yr)

$^{2}\mathrm{D} + {}^{1}\mathrm{H} \rightarrow {}^{3}\mathrm{He} + \gamma$ (5.493 MeV, 6 s)

pp I chain

$^{3}\mathrm{He} + {}^{3}\mathrm{He} \rightarrow {}^{4}\mathrm{He} + {}^{1}\mathrm{H} + {}^{1}\mathrm{H}$ (12.85 MeV, 10^{6} yr)

Note: net $6\ {}^{1}\mathrm{H} \rightarrow {}^{4}\mathrm{He} + 2\ {}^{1}\mathrm{H}$

pp II chain

$^{3}\mathrm{He} + {}^{4}\mathrm{He} \rightarrow {}^{7}\mathrm{Be} + \gamma$ (1.586 MeV)

$^{7}\mathrm{Be} + e^{-} \rightarrow {}^{7}\mathrm{Li} + \nu_e$ (0.861 MeV)

$^{7}\mathrm{Li} + {}^{1}\mathrm{H} \rightarrow {}^{4}\mathrm{He} + {}^{4}\mathrm{He}$ (17.347 MeV)

pp III chain

$^{3}\mathrm{He} + {}^{4}\mathrm{He} \rightarrow {}^{7}\mathrm{Be} + \gamma$

$^{7}\mathrm{Be} + {}^{1}\mathrm{H} \rightarrow {}^{8}\mathrm{B} + \gamma$ (0.135 MeV)

$^{8}\mathrm{B} +$ - , $^{8}\mathrm{Be} + e^{+} + \nu_e$ *This neutrino carries away 7.2 MeV*

$^{8}\mathrm{Be} \rightarrow {}^{4}\mathrm{He} + {}^{4}\mathrm{He}$ (18.074 MeV)

0.420 MeV to the positron and neutrino (0.26 MeV); position and electron (each 0.511 MeV rest energy) annihilate ➔ 1.442 MeV released

pp I important when $T_c > 5 \times 10^{6}\mathrm{K}$

$$Q_{total} = 1.44 \times 2 + 5.49 \times 2 + 12.85 = 26.7\ \mathrm{MeV}$$

$$Q_{net} = 26.7 - 0.26 \times 2 = 26.2\ \mathrm{MeV}$$

$$\rightarrow 6 \times 10^{18}\ \mathrm{erg\ g^{-1}}$$

- ✓ The baryon number, lepton number, and charges should all be conserved.
- ✓ All 3 branches operate simultaneously.
- ✓ pp I is responsible for > 90% of stellar luminosity

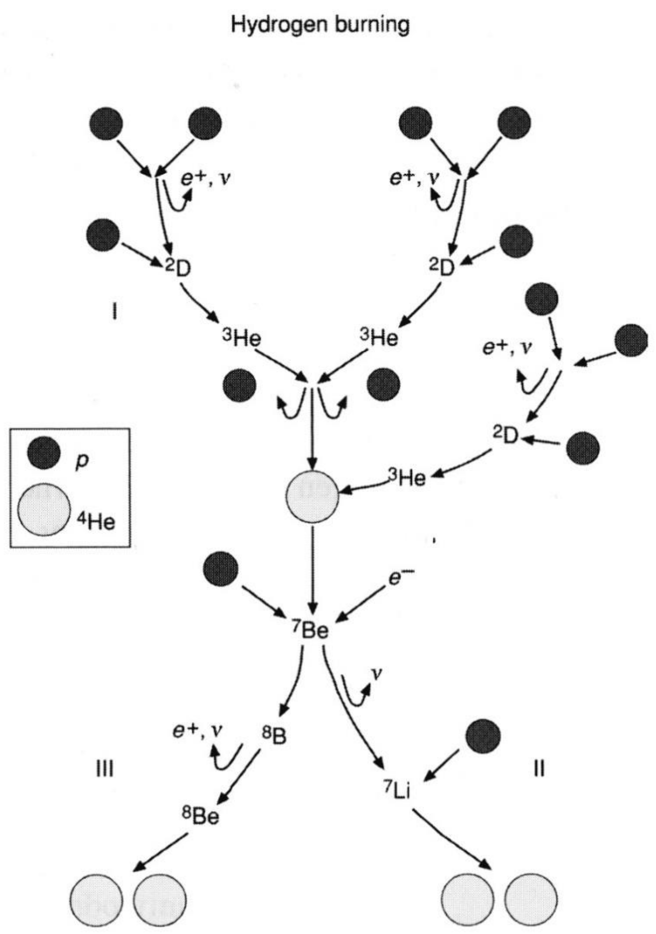


Figure 4.3 The nuclear reactions of the p – p I, II and III chains.

Among all fusion processes, the p-p chain has the lower temperature threshold, and the weakest temperature dependence.

$$Q_{pp} = (M_{4H} - M_{He})\, c^2$$

$$= 26.731 \text{ MeV}$$

$(M_{4H} - M_{He})$: **mass deficit**

But some energy (up to a few MeV, depending on the reactions) is carried away by neutrinos.

… but the nucleus of deuterium, a deuteron, consists of a proton and a neutron!

- ✓ $p + p \rightarrow {}^{2}He$ (unstable)$\rightarrow p + p$
- ✓ Hans Bethe (1939) realized that the weak interaction was capable of converting a proton to a neutron (!) first
- ✓ Weak interaction → a very small cross section
- ✓ The neutron is more massive, so this requires energy, i.e., it is an endothermic process, but neutron + proton → deuteron (releasing binding energy, i.e., is exothermic)

The thermonuclear reaction rate is

$$r_{pp} = 3.09 \times 10^{-37}\ n_p^2\ T_6^{-2/3} \exp\left(-33.81\, T_6^{-1/3}\right)$$

$$(1 + 0.0123\, T_6^{1/3} + 0.0109\, T_6^{2/3} + 0.0009\, T_6)\quad [\mathrm{cm^{-3} s^{-1}}],$$

where the factor $3.09 \times 10^{-37} n_p^2 = 11.05 \times 10^{10}\ \rho^2 X_H^2$

And the energy generation rate is

$$q_{pp} = 2.38 \times 10^{6}\ \rho\ X_H^2\ T_6^{-2/3} \exp\left(-33.81\, T_6^{-1/3}\right)$$

$$(1 + 0.0123\, T_6^{1/3} + 0.0109\, T_6^{2/3} + 0.0009\, T_6)\quad [\mathrm{erg\ g^{-1} s^{-1}}]$$

- PP I vs PP II
  That is, ^{3}He to react with ^{3}He at a lower temperature,

  or to react with ^{4}He at $\boldsymbol{T > 1.4 \times 10^7}$ **K**

- Relative importance of each chain
  $\longrightarrow$ Branching ratio $\longleftrightarrow$ T, ρ, μ

- Above $T > 3 \times 10^7$ K, PP III should dominate, but in reality, at this temperature, other (CNO) reactions take over.

- The overall rate of energy generation is determined by the slowest reaction, i.e., the first one, with reaction time 10^{10} yrs

  $Q_{pp} \sim 26.73$ MeV (≈ 6.54 MeV per proton)

  $q_{pp} \sim \rho^1 T^n, n \sim 4 - 6$

$n \sim 6$ for T $\approx 5 \times 10^6$ K $n \sim 3.8$ for T $\approx 15 \times 10^6$ K (Sun) $n \sim 3.5$ for T $\approx 20 \times 10^6$ K

CNO cycle (bi-cycle)   C, N, O as catalysts

$^{12}C + {^1H} \rightarrow {^{13}N} + \gamma$ 10^6 y	$^{14}N + {^1H} \rightarrow {^{15}O} + \gamma$
$^{13}N \rightarrow {^{13}C} + e^+ + \nu$ 14 min	$^{15}O \rightarrow {^{15}N} + e^+ + \nu$
$^{13}C + {^1H} \rightarrow {^{14}N} + \gamma$ 3×10^5 y	$^{15}N + {^1H} \rightarrow {^{16}O} + \gamma$ ←
$^{14}N + {^1H} \rightarrow {^{15}O} + \gamma$ 3×10^8 y	$^{16}O + {^1H} \rightarrow {^{17}F} + \gamma$
$^{15}O \rightarrow {^{15}N} + e^+ + \nu$ 82 s	$^{17}F \rightarrow {^{17}O} + e^+ + \nu$
→ $^{15}N + {^1H} \rightarrow {^{12}C} + {^4He}$ 10^4 y	$^{17}O + {^1H} \rightarrow {^{14}N} + {^4He}$

Recognized by Bethe and independently by von Weizsäcker

CN cycle + NO cycle

Cycle can start from any reaction as long as the involved isotope is present.

CN cycle more significant

NO cycle efficient only when $T > 20 \times 10^6$ K

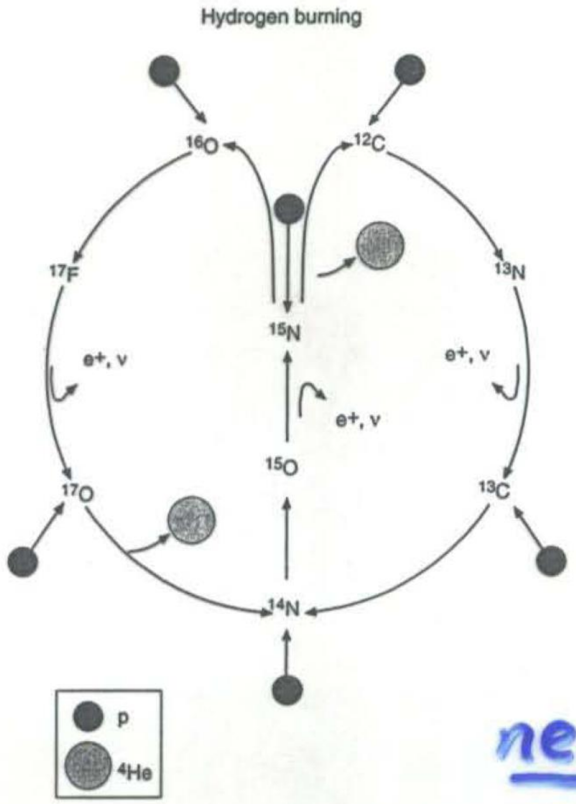


Figure 4.4 The nuclear reactions of the CNO bi-cycle.

net

$4H \rightarrow {^4He}$

$Q_{CNO} \sim 25$ MeV   after that carried away by the neutrinos

$\varepsilon_{CNO} \sim \rho T^{16}$

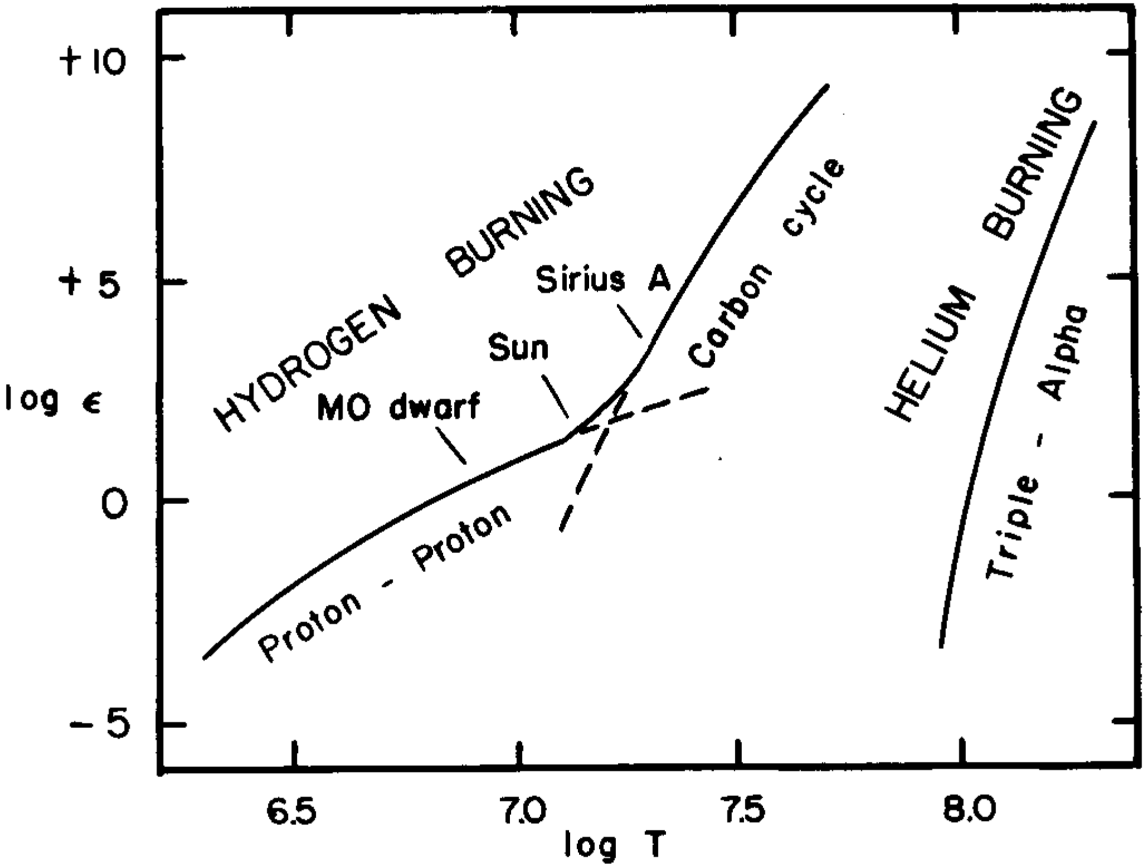


Fig. 10.1. Nuclear energy generation as a function of temperature (with $\rho X^2 = 100$ and $X_{CN} = 0.005X$ for the proton-proton reaction and the carbon cycle, but $\rho^2 Y^3 = 10^8$ for the triple-alpha process).

Schwarzschild

- At the center of the Sun, $q_{CNO}/q_{pp} \approx 0.1$
- CNO dominates in stars $> 1.2\ M_\odot$, i.e., of a spectral type F7 or earlier
  → large energy outflux
  ➔ a convective core
- This separates the lower and upper MS.

✓ CN cycle takes over the PP chains near $T_6 = 18$.
✓ Helium burning starts ~10^8 K.

A He Gas — the triple-alpha process

He-burning ignites at Tc ~ 10^8 K

^{4}He + ^{4}He → ^{8}Be (−95 keV, i.e., endothermic)

The lifetime of ^{8}Be is 2.6×10^{-16} s but is still longer than the mean-free time between α particles at T_8 (Edwin Salpeter, 1952)

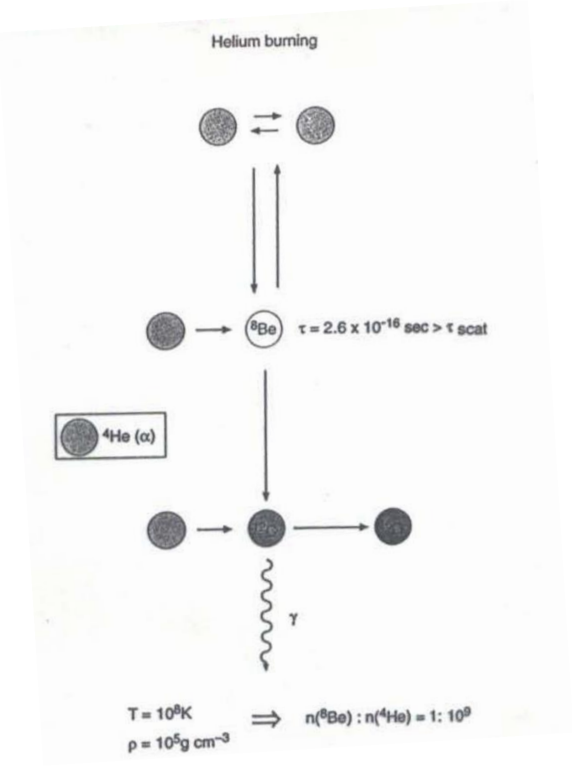
^{8}Be + ^{4}He → ^{12}C + γ (7.4 MeV) ← bottleneck

Note: net 3 ^{4}He → ^{12}C

$Q_{3\alpha} = 7.275$ MeV ↪ 5.8×10^{17} erg g^{-1} ~ 0.1 of H → He

net: 3 ^{4}He → ^{12}C

$\varepsilon_{3\alpha} \sim \rho^2 T^{40}$ ∵ bottleneck = 2nd reaction ↔ ^{8}Be



Nucleosynthesis during helium burning

$C^{12}(\alpha,\gamma)O^{16}$, $Q = 7.162$ MeV

$O^{12}(\alpha,\gamma)Ne^{16}$

A succession of (α,γ) processes → $^{16}O, ^{20}Ne, ^{24}Mg$ … (the α-process)

A carbon/oxygen Gas

$^{12}C + ^{12}C$

$T \gtrsim 5\times10^{8}$ K

$Q_{cc} \sim 13$ MeV

$$^{12}C + ^{12}C \to ^{24}Mg + \gamma$$
$$\to ^{23}Mg + n$$
$$\to ^{23}Na + p$$
$$\to ^{20}Ne + \alpha$$
$$\to ^{16}O + 2\alpha$$

$$^{16}O + ^{16}O \to ^{32}S + \gamma$$
$$\to ^{31}S + n$$
$$\to ^{31}P + p$$
$$\to ^{28}Si + \alpha$$
$$\to ^{24}Mg + 2\alpha$$

$^{16}O + ^{16}O$

$T \gtrsim 10^{9}$ K

$Q_{OO} \sim 16$ MeV

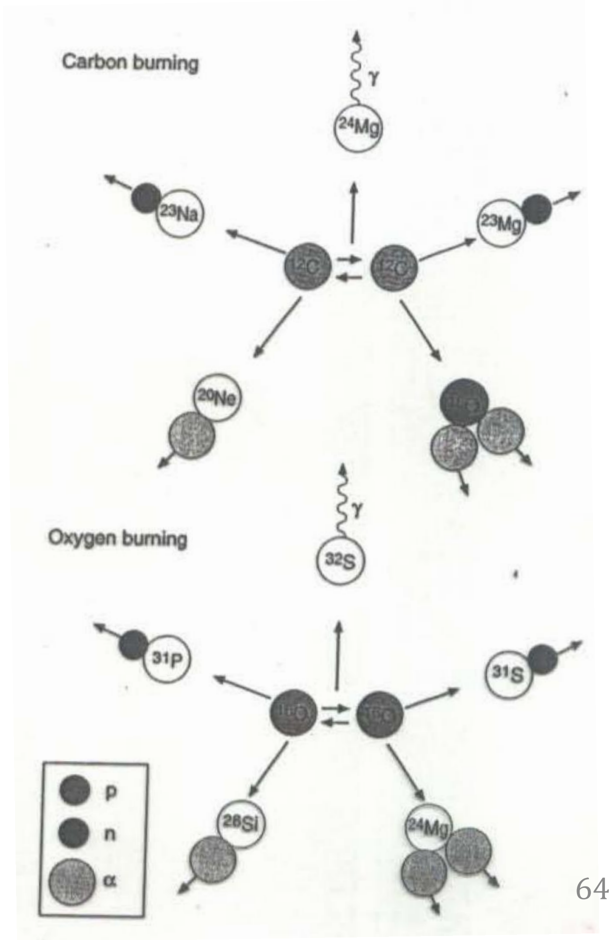
C-burning ignites when Tc ~ (0.3-1.2) × 10^9 K, i.e., for stars 15-30 $M_\odot$

O-burning ignites when Tc ~ (1.5-2.6) × 10^9 K, i.e., for stars > 15-30 $M_\odot$

The p and α particles produced are captured immediately (because of the low Coulomb barriers) by heavy elements

➔ isotopes

O burning → Si



$$q_{PP} = 2.4 \times 10^6 \, \rho \, X^2 \, T_6^{-2/3} \exp\left[-33.8 \, T_6^{-1/3}\right] \quad [\mathrm{erg\,g^{-1}\,s^{-1}}]$$

$$q \propto \rho \, X_H^2 \; T^{4}$$

$$q_{CN} = 8 \times 10^{27} \, \rho \, X \, X_{CN} \, T_6^{-2/3} \exp\left[-152.3 \, T_6^{-1/3}\right] \quad [\mathrm{erg\,g^{-1}\,s^{-1}}]$$

$$q \propto \rho \, X_H X_{CN} \, T^{16}$$

$$\frac{X_{CN}}{X_H} = 0.02 \text{ ok for Pop I}$$

$$q_{3\alpha} = 3.9 \times 10^{11} \, \rho^2 X_\alpha^{\,3} \, T_8^{-3} \exp[-42.9 \, T_8] \quad [\mathrm{erg\,g^{-1}\,s^{-1}}]$$
$$\approx 4.4 \times 10^{-8} \, \rho^2 X_\alpha^{\,3} \, T_8^{\,40} \quad [\mathrm{erg\,g^{-1}\,s^{-1}}] \quad (\text{if } T_8 \approx 1)$$

Clayton

Does ^{28}Si follow the same scenario?

$^{28}Si + ^{28}Si \xrightarrow[?]{} {}^{56}Fe$?

C+C 6×10^{8} K

O+O 10^{9} K

No! Coulomb barrier becomes extremly high; another nuclear reaction takes place

eg.

Photoionization

Likewise

Photodisintegration

For example, $^{16}\mathrm{O} + \alpha \leftrightarrow {}^{20}\mathrm{Ne} + \gamma$

If $T < 10^9$ K $\rightarrow$

but if $T \geq 1.5 \times 10^9$ K (in radiation field) $\leftarrow$

So $^{28}\mathrm{Si}$ disintegrates at $\approx 3 \times 10^9$ K to lighter elements

(then recaptured ...)

until a nuclear statistical equilibrium is reached

But the equilibrium is not exact

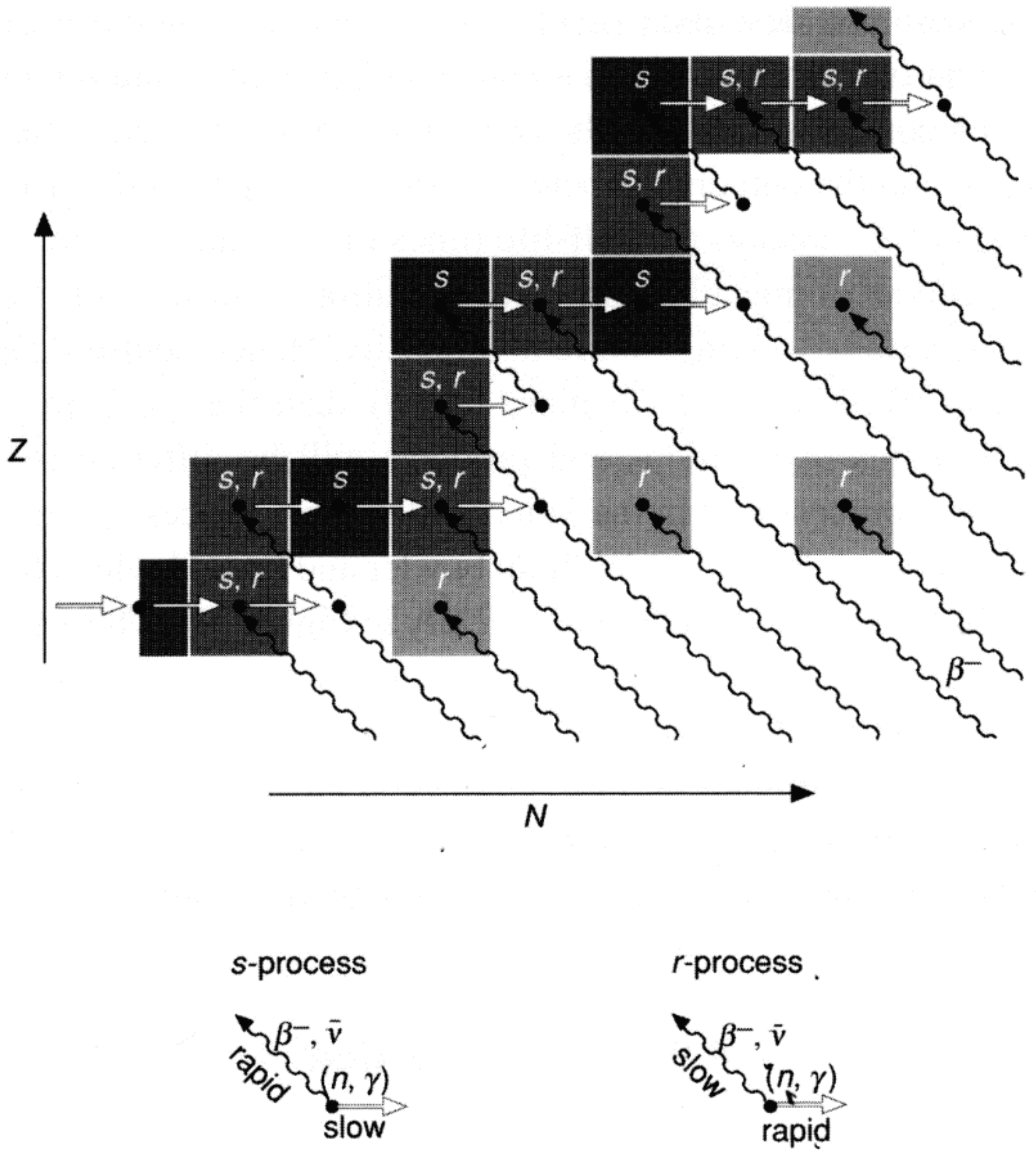
→ a pileup of the iron group nuclei (Fe, Co, Ni)

which can resist photodisintegration until 7×10^9 K

Nuclear Fuel	Process	$T_{threshold}$ (10^6 K)	Products	Energy per nucleon (MeV)
H	p-p	~4	He	6.55
H	CNO	15	He	6.25
He	3α	100	C, O	0.61
C	C + C	600	O, Ne, Na, Mg	0.54
O	O + O	1,000	Mg, S, P, Si	~0.3
Si	Nuc. Equil.	3,000	Co, Fe, Ni	<0.18

From Prialnik Table 4.1

- Interactions among charged particles $\longrightarrow$ Coulomb barrier
- If there are enough neutrons around $\longrightarrow$ neutron capture, not limited by Coulomb barrier, so proceed at relatively low Ts
  $\longrightarrow$ ever heavier isotopes or
  $\longrightarrow$ radioactive decay
  → a new element $+e^-$ (beta decay) $+ \bar{\nu}$ (antineutrino)
- Stable nuclei: neutron captures
- Unstable nuclei: neutron capture or β^- decay
- β^- decay has a constant time scales
- n^0 capture time scales $\longleftrightarrow$ (T, ρ), so may proceed slower (***s*-process**) or more rapidly (***r*-process**) than the competing β^- decays



Prialnik Fig. 4.7

- Nuclear reactions: mass to energy (light)
- The reverse, energy into mass, is also possible; e.g., a photo → an electron + a position, if $h\nu > 2m_e c^2$, with the presence of a nucleus
- $kT \approx h\nu \approx 2m_e c^2, T \approx 1.2 \times 10^{10}$ K
- In reality, at $T \gtrsim 10^9$ K, sufficient photons (tail of the Planck function) for **pair production. Annihilation** immediately destroys the positrons.

$$^{56}Fe + 124 \text{ MeV} \rightarrow 13\ ^{4}He + 4\,n$$

If $T \uparrow\uparrow\uparrow$, even $^{4}He \rightarrow p^{+} + n^{0}$

So stellar interior has to be between a few T_6 and a few T_9.

Lesson: Nuclear reactions that absorb (rather than emit) energy from ambient radiation field (in stellar interior) can lead to catastrophic consequences.

Luminosity

Ohm's law in a circuit, $I = V / R$,

in electromagnetics, $\vec{J}$ [current density] = σ [conductivity] $\vec{E}$ [electric field]

In hydraulics, [flow] $\propto$ [pressure gradient] / [resistance]

$$L \sim 4\pi R^2 \frac{d\left(\frac{1}{3}aT^4\right)/dr}{\kappa\rho}$$

$$\sim 4\pi R^2 \frac{4}{3}\frac{aT^3}{\kappa\rho}\frac{dT}{dr}$$

$$\sim \frac{R^2T^3}{\kappa\rho}\frac{dT}{dr}$$

(unit) Pressure = [energy] / [volume]

Blackbody radiation

Energy density $u = aT^4$

Radiation pressure $P_{\mathrm{rad}} = (1/3)\,u$

$$\frac{dT(r)}{dr} = -\frac{3\kappa\rho L(r)}{4ac\,4\pi r^2 T^3}$$

For a given structure,

$$T \sim T_c, \frac{dT}{dr} \sim \frac{T_c}{R}, T_c \sim \frac{\mu G M}{R}$$

$$L \sim \frac{R^2\, T^4/R}{\kappa(M/R^3)} \sim \frac{R^4 T^4}{\kappa M} \sim \frac{R^4}{\kappa M}\left(\frac{\mu G M}{R}\right)^4$$

$$\boxed{L \sim \frac{\mu^4 \mathrm{G}^4 \mathrm{M}^3}{\kappa}}$$

The opacity $\kappa = \kappa(\rho, T, \mu)$

$$L \sim \frac{\mu^4 G^4 M^3}{\kappa}$$

$$T \sim \frac{\mu GM}{R}$$

- For solar composision, Kramers opacity

$$\kappa \sim \rho T^{-3.5} \quad \text{valid for } 10^4\text{–}10^6 \text{ K.}$$

So $\kappa \sim \mu^{-3.5} G^{-3.5} M^{-2.5} R^{0.5}$

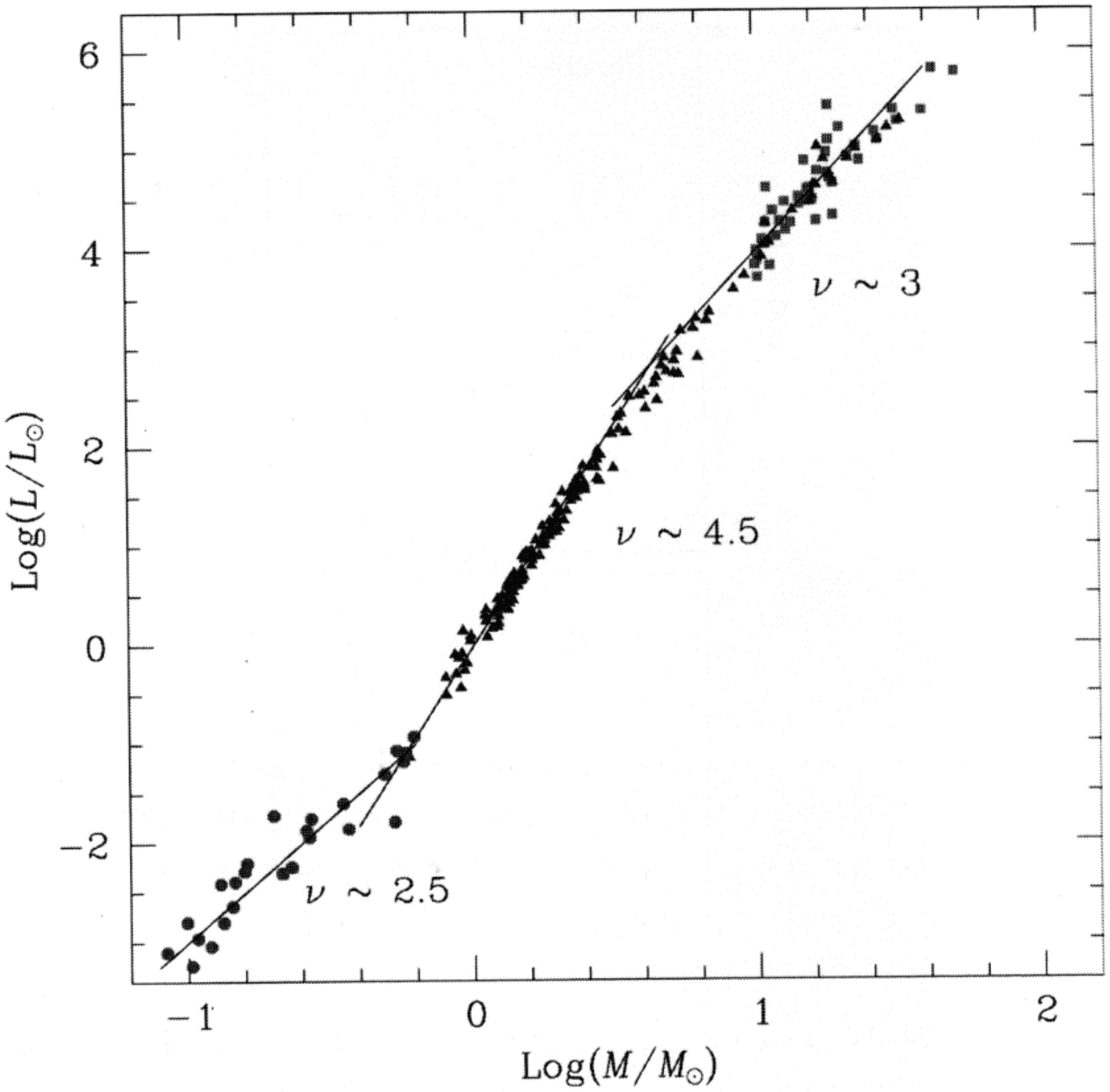
and $\boxed{L \sim \mu^{7.5} G^{7.5} M^{5.5} R^{-0.5}}$

- For high-mass stars, i.e., high temperature and low density, opacity by electron scattering

$$\kappa = 0.2\,(1 + X)\ \text{cm}^2\text{g}^{-1} = \text{const.}$$

and $\boxed{L \sim \mu^4 G^4 M^3}$

Mass-luminosity relation for main-sequence stars



$$\frac{L}{L_\odot} = \left(\frac{M}{M_\odot}\right)^\nu$$

Prianik Fig. 1.6

$$T_c \approx \frac{\mu\, GM}{R}$$

$$\left.\begin{array}{r} \text{So for a given } T_c, M \;\rightarrow R \\ \text{MLR} \;\rightarrow L \end{array}\right\} L\,(\propto R^2 T_{\text{eff}}^4) \text{ and } T_{\text{eff}}$$

Main sequence is a run of L and T_{eff} as a function of stellar mass, with T_c nearly constant.

Why $T_c \approx$ constant?
Because onset of H burning $\sim 10^7$ K regardless of the stellar mass

The main sequence

Recall for low-mass stars, $L \propto M^{5.5}\, R^{-0.5}$, pp chain $q \propto \rho_{\mathrm{c}}\, T^4$

The energy-generation equation,

$$\frac{dL}{dr} = 4\pi r^2 \rho_c\, q$$

$$\Longrightarrow L \propto R^3 \rho_c^2\, T^4 = R^3 \left(\frac{M}{R^3}\right)^2 \left(\frac{M}{R}\right)^4 = \frac{M^6}{R^7}$$

$R \sim M^{1/13}$ .... Stellar radius ↔ very weakly on the mass

$L \sim M^{71/13} \approx M^{5.5}$... Stellar Luminosity ↔ strongly on the mass

The main sequence in the HRD

Recall for <u>low-mass stars</u>, $L \propto M^{5.5} R^{-0.5}$, pp chain $q \propto \rho_c T^4$

The energy-generation equation,

$$\frac{dL}{dr} = 4\pi r^2 \rho_c\, q$$

$$\Longrightarrow L \propto R^3 \rho_c^2\, T^4 = R^3 \left(\frac{M}{R^3}\right)^2 \left(\frac{M}{R}\right)^4 = \frac{M^6}{R^7}$$

$R \sim M^{1/13}$ .... Stellar radius varies weakly with the mass

$L \sim M^{71/13} \approx M^{5.5}$... Stellar Luminosity varies strongly ...

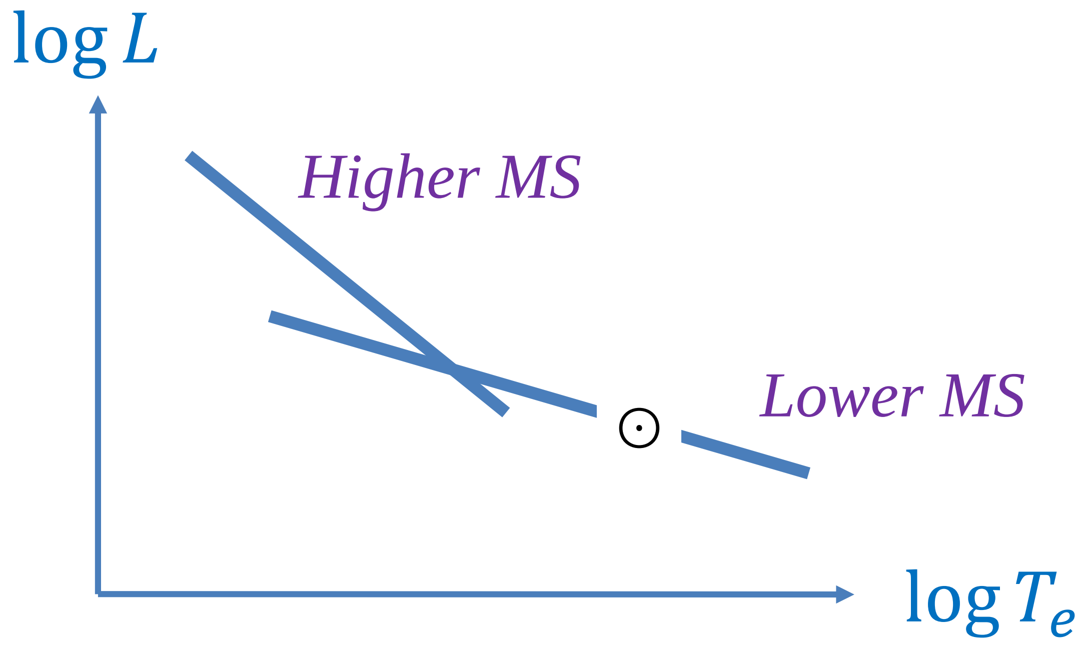
In the HRD, $L \propto R^2\, T_e^4 \longrightarrow L^{981/1007} \propto T_e^4$

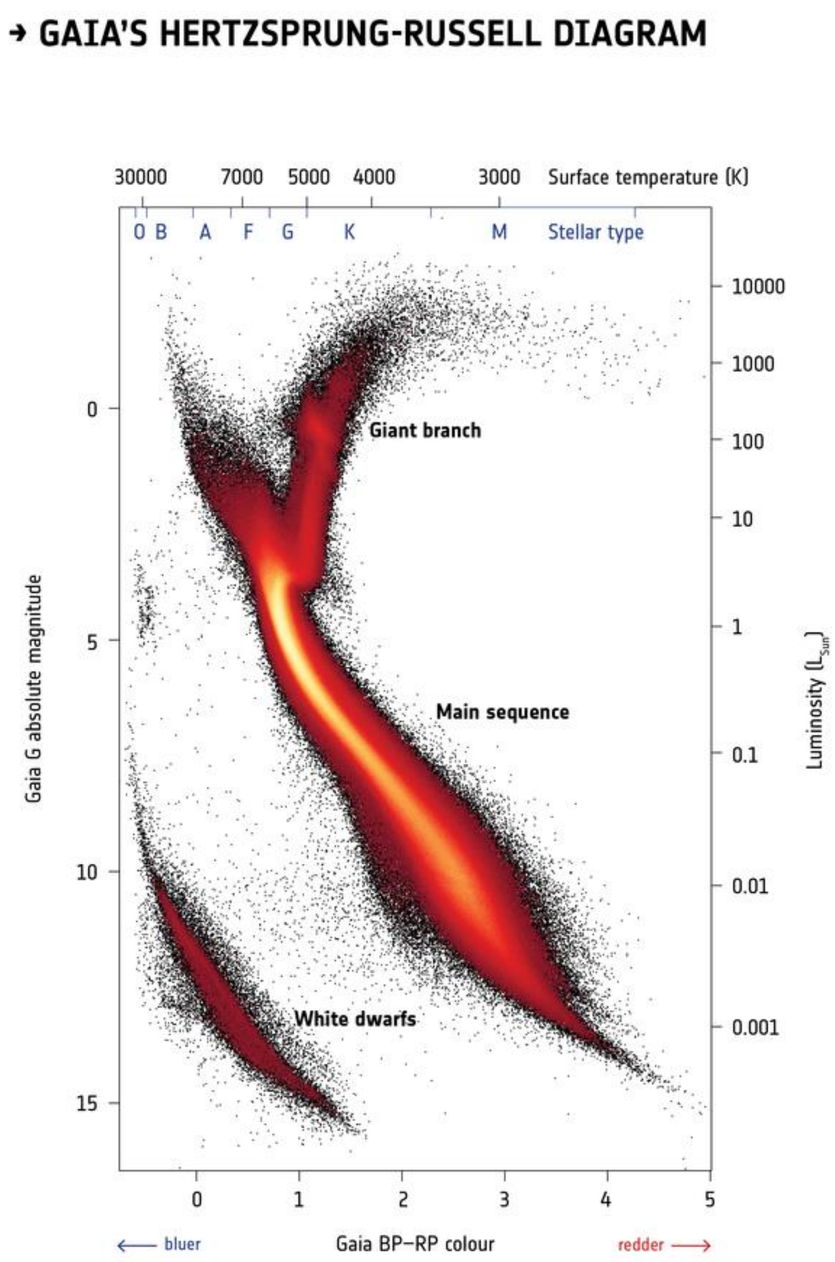
or $\log L \approx 4 \log T_e +$ const (i.e., constant radius)

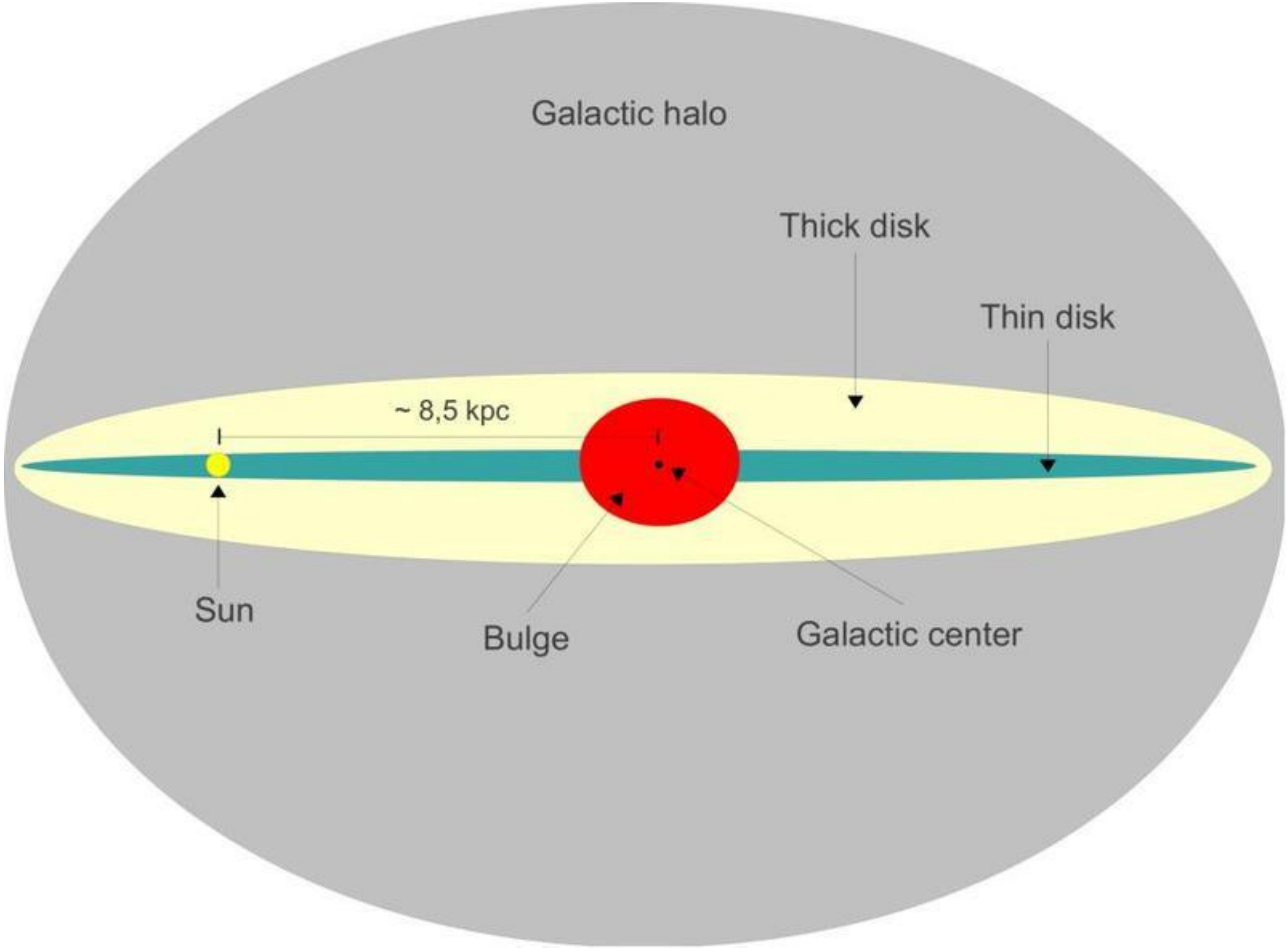
For <u>high-mass stars</u>, $L \propto M^3$,
CNO cycle $q \propto \rho_c T^{16}$

Then, $M^{15} \propto R^{19}$, so $L \propto T_e^{76/9}$
or $\log L \approx 8.4 \log T_e + \text{const}$

That is, a steeper MS slope in the HRD



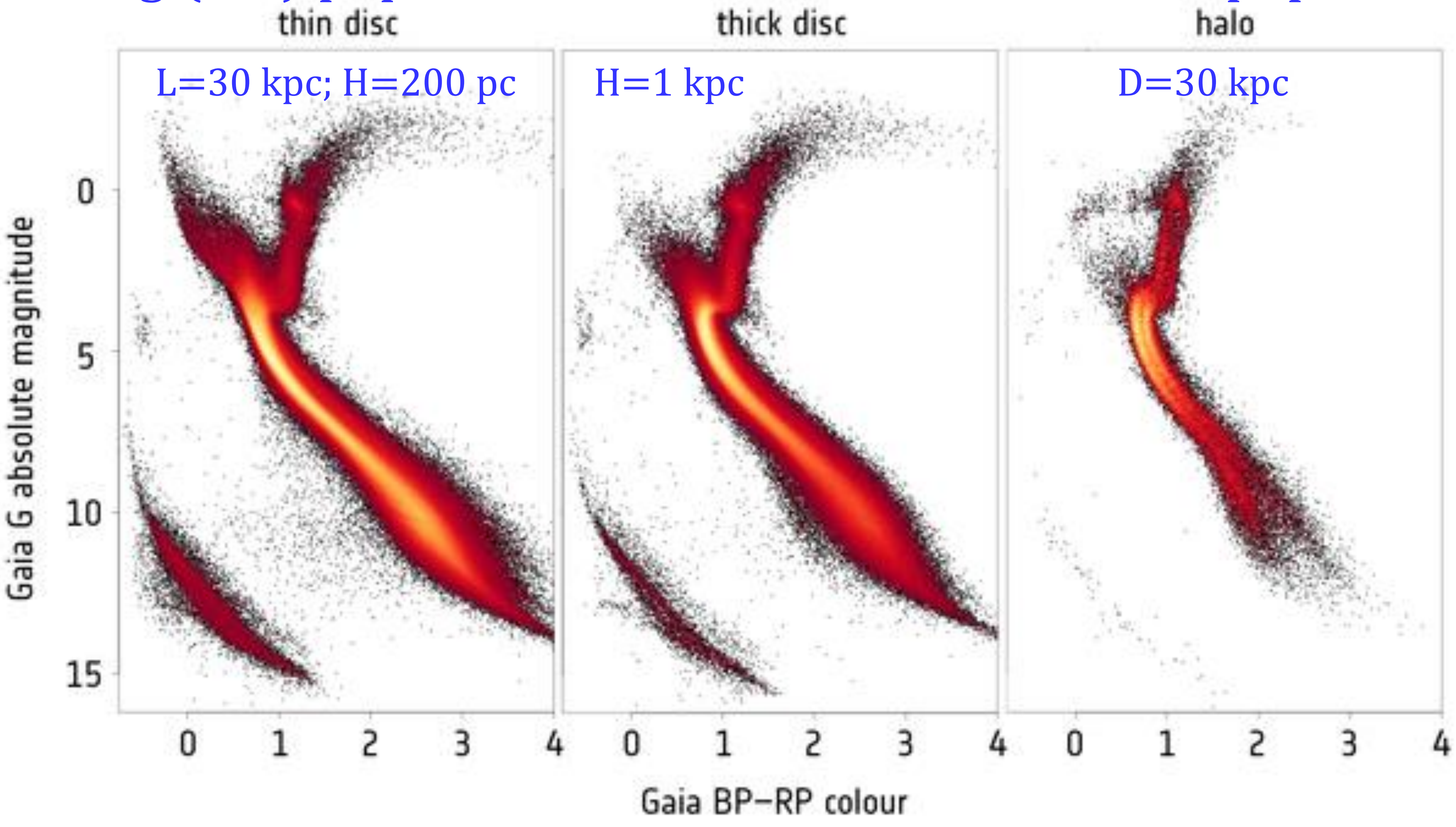



Galactic halo
Thick disk
Thin disk
~ 8,5 kpc
Sun
Bulge
Galactic center

Metal poorer

Young (hot) population

Bimodal population?



https://sci.esa.int/web/gaia/-/60210-gaia-s-hertzsprung-russell-diagram-for-different-populations-of-stars

Main Sequence Lifetimes

$$\tau_{\text{Nuclear}} \propto \frac{M}{L}$$

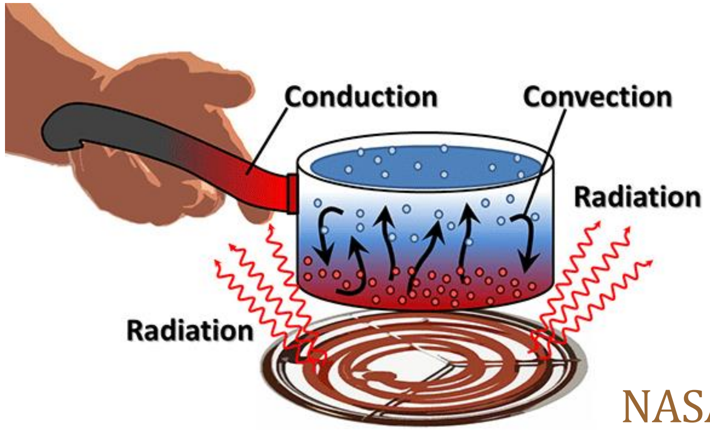
$\propto M^{-4.5}$ (for low-mass stars)

or

$\propto M^{-2}$ (for massive stars)

Calibrated with the Sun.

Energy can be transported by conduction or convection, or radiation.



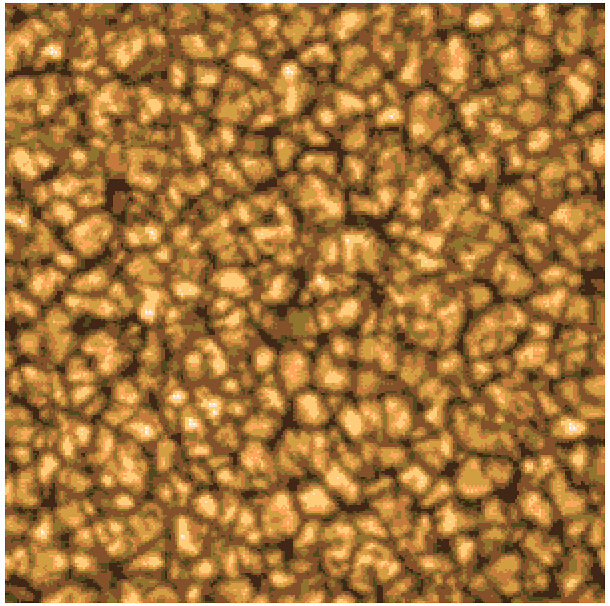
NASA

<u>Conduction</u>: by microscopic collision of particles and movement of electrons.
Flux density [erg/s/cm^2] $= -\kappa \nabla T$

<u>Convection</u>: by bulk motion of particles in a fluid (gas or liquid): *advection*（平流）(directional flow of energy) or *diffusion*（擴散）(non-directional along a concentration gradient).

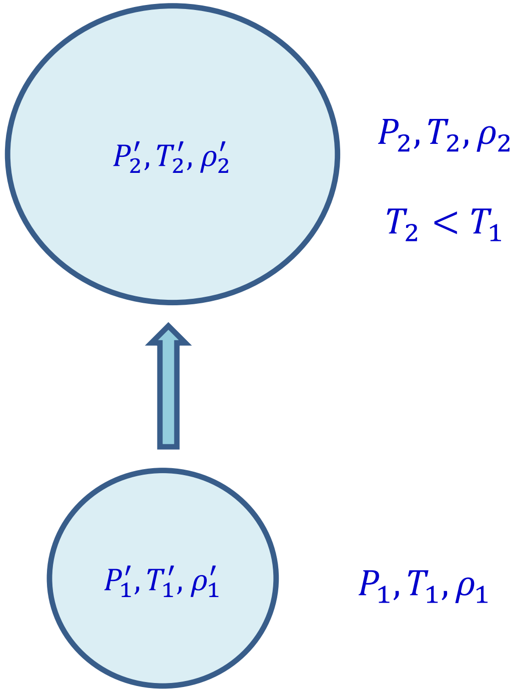
Convection does not happen in solids.

Stars transport energy by either radiation or convection. Conduction is effective only in compact objects, e.g., in isothermal cores in WDs.



Convective equilibrium (stability vs instability)

Convection takes over? When an element moves vertically, does it continue to move? Key: Temperature gradients



Element maintaining pressure equilibrium with surrounding, $P_2' = P_2$, ideal gas law $\rightarrow \rho_2 T_2 = \rho_2' T_2'$,

Consider an element floats upwards

If $\rho_2' > \rho_2$ (or $T_2' < T_2$) $\rightarrow$ sink back; no convection

To <u>have</u> convection, the element (rising adiabatically) should cool slower than the surrounding (in radiative equilibrium), i.e.,

$$\left(\frac{dT}{dr}\right)_{\text{element}} < \left(\frac{dT}{dr}\right)_{\text{surrounding}} \quad \text{or} \quad \left(\frac{dT}{dr}\right)_{\text{ad}} < \left(\frac{dT}{dr}\right)_{\text{rad}}$$

⇒ Convection sets in when the adiabatic temp. gradient is smaller than temp. gradient by radiative equil.

i.e.,
$$\left(\frac{dT}{dr}\right)_{ad} < \left(\frac{dT}{dr}\right)_{rad}$$

Compared with the surrounding temperature gradient

Radiation can no longer transport the energy efficiently enough
➔ Convective instability

For an adiabatic process, $PV^{\gamma} = constant$

The rising height is typified by the mixing length ℓ, or parameterized as the scale height H, defined as the pressure (or density) varies by a factor of e times. Usually $0.5 \lesssim \ell/H \lesssim 2.0$

Since $\frac{dP}{dr} = -\rho g$ and $P = \rho k T$

$dT \cdot \frac{dP}{dr}\frac{1}{P} \propto \frac{1}{T} \cdot dT$

$\therefore \frac{dT}{dr} \propto \frac{dT/T}{dP/P} = \frac{d\ln T}{d\ln P}$

$\Rightarrow$ Criterion for convection equilibrium becomes

$$\left(\frac{d\ln T}{d\ln P}\right)_{ad} < \left(\frac{d\ln T}{d\ln P}\right)_{rad}$$

With the notation ∇ (nabla)

$$\nabla_{ad} < \nabla_{rad}$$

Convection takes place when the temperature gradient is "sufficiently" high (compared with the adiabatic condition) or the pressure gradient is low enough.

$$\left(\frac{dT}{dr}\right)_{\mathrm{ad}} < \left(\frac{dT}{dr}\right)_{\mathrm{rad}}$$

$$\left(\frac{d\ln T}{d\ln P}\right)_{\mathrm{ad}} < \left(\frac{d\ln T}{d\ln P}\right)_{\mathrm{rad}}$$

Such condition also exists when the gas absorbs a great deal of energy without temperature increase, e.g., with phase change or ionization

→ when c_V is large or γ is small

$$\gamma = \frac{Nk}{c_V} + 1$$

In meteorology, dry and cool air tends to be stable, whereas wet and warm air (smaller gamma values) is vulnerable to convection → thunderstorm

How to calculate ∇_{rad}?

$$\frac{dT}{dr} = -\frac{3}{4ac}\frac{\kappa\rho}{T^3}\frac{L(r)}{4\pi r^2} \text{ and } \frac{dP}{dr} = -g\rho$$

So $\frac{dT}{dP} \propto \frac{\kappa}{T^3}\frac{L(r)}{r^2}$

$$\nabla_{\mathrm{rad}} \equiv \left(\frac{d\ \ln T}{d\ \ln P}\right)_{\mathrm{rad}} = \frac{dT/T}{dP/P} = \cdots = \frac{3\kappa}{16\pi \mathrm{ac}}\frac{P}{T^4}\frac{L(r)}{GM(r)}$$

Note that for an adiabatic process for an ideal gas

- $P = nkT \propto \rho T$

So $\frac{dP}{P} = \frac{d\rho}{\rho} + \frac{dT}{T}$

And recall again

- ✓ $nk = c_p - c_v$
- ✓ $\gamma = \frac{c_p}{c_v} = \frac{1+n/2}{n/2} = 1 + \frac{2}{n}$, where n is d.o.f.
- ✓ Note $n \nearrow, \gamma \searrow$

How to calculate $\nabla_{\rm ad}$?

$$dQ = c_v dT + Pd\left(\frac{1}{\rho}\right) = c_v dT - \frac{\mathrm{P}}{\rho^2} d\rho = 0$$

$$c_v dT = \frac{P}{\rho^2} d(\rho) \rightarrow c_v \frac{dT}{T} = \frac{P}{\rho T} \frac{d\rho}{\rho} \rightarrow c_v \frac{dT}{T} = (c_p - c_v)\left(\frac{dP}{P} - \frac{dT}{T}\right)$$

$$\Longrightarrow c_p \frac{dT}{T} = (c_p - c_v) \frac{dP}{P}$$

$$\nabla_{\rm ad} \equiv \left(\frac{d\ \ln T}{d\ \ln P}\right)_{\rm ad} = \frac{dT/_T}{dP/_P} = 1 - \frac{c_v}{c_p} = 1 - \frac{1}{\gamma} = 0.4 \text{ for a monatomic gas}$$

for which $\gamma = 5/3$.

Note $\gamma \searrow, \nabla_{\rm ad} \searrow$

So the condition for convective instability (convection to take place) is $\left(\frac{d \log T}{d \log P}\right) < 0.4$.

Note. $\nabla_{rad} \propto P$

At surface $\nabla_{rad} \to 0$

$\therefore \quad \nabla_{ad} \overset{\text{always}}{>} \nabla_{rad} \Rightarrow$ no convection!

The outermost layers of a star are always in radiative equilibrium.

$\therefore$ Convection occurs either

① large temperature gradient for radiative equilibrium

② small adiabatic temperature gradient

Convection occurs when $\nabla_{\mathrm{rad}} > \nabla_{\mathrm{ad}}$

That is, when ∇_{rad} is large, or when ∇_{ad} is small.

To recap

$$\nabla_{\mathrm{rad}} = \frac{dT}{dr} = \frac{L_r}{r^2}\frac{\kappa\rho}{\sigma T^3}$$

$$\nabla_{\mathrm{ad}} = 1 - \frac{1}{\gamma}, \text{ where } \gamma = {}^{c_p}/_{c_v}$$

→ ∇_{ad} small → c_v large ➔ H_2 dissociation (PMS Hayashi tracks)
H ionization, T~6,000 K
He ionization, T~20,000 K
He II ionization, T~50,000 K

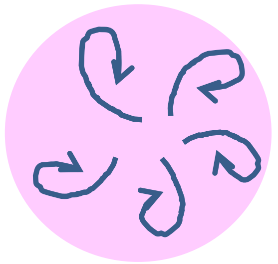
Ionization satisfies <u>both</u> conditions because

1. Opacity ↑
2. e^- receive energy ⟶ d.o.f. ↗, so γ ↘ ⟶ ∇_{ad} ↘
   ⇒ susceptable to convection

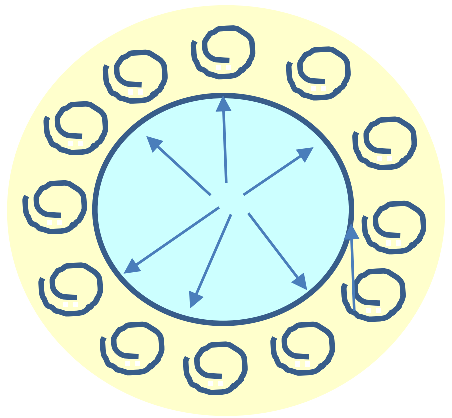
➔ Development of hydrogen convective zones inside stars.

Similarly, there are 1st and 2nd helium convective zones.

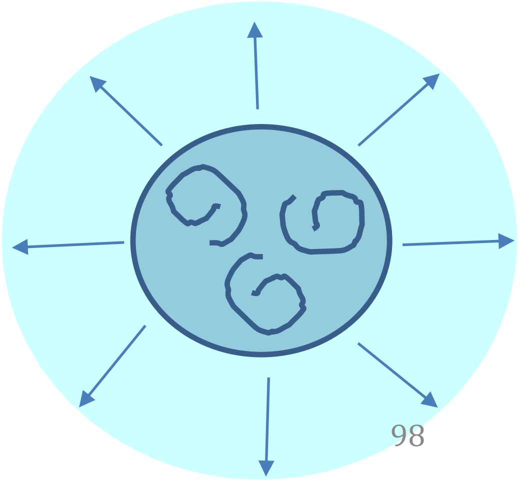
For a **very low-mass star** $\left(M \lesssim 0.4\ M_{\odot}\right)$, ionization of H and He leads to a fully convective star ➔ H completely burns off.

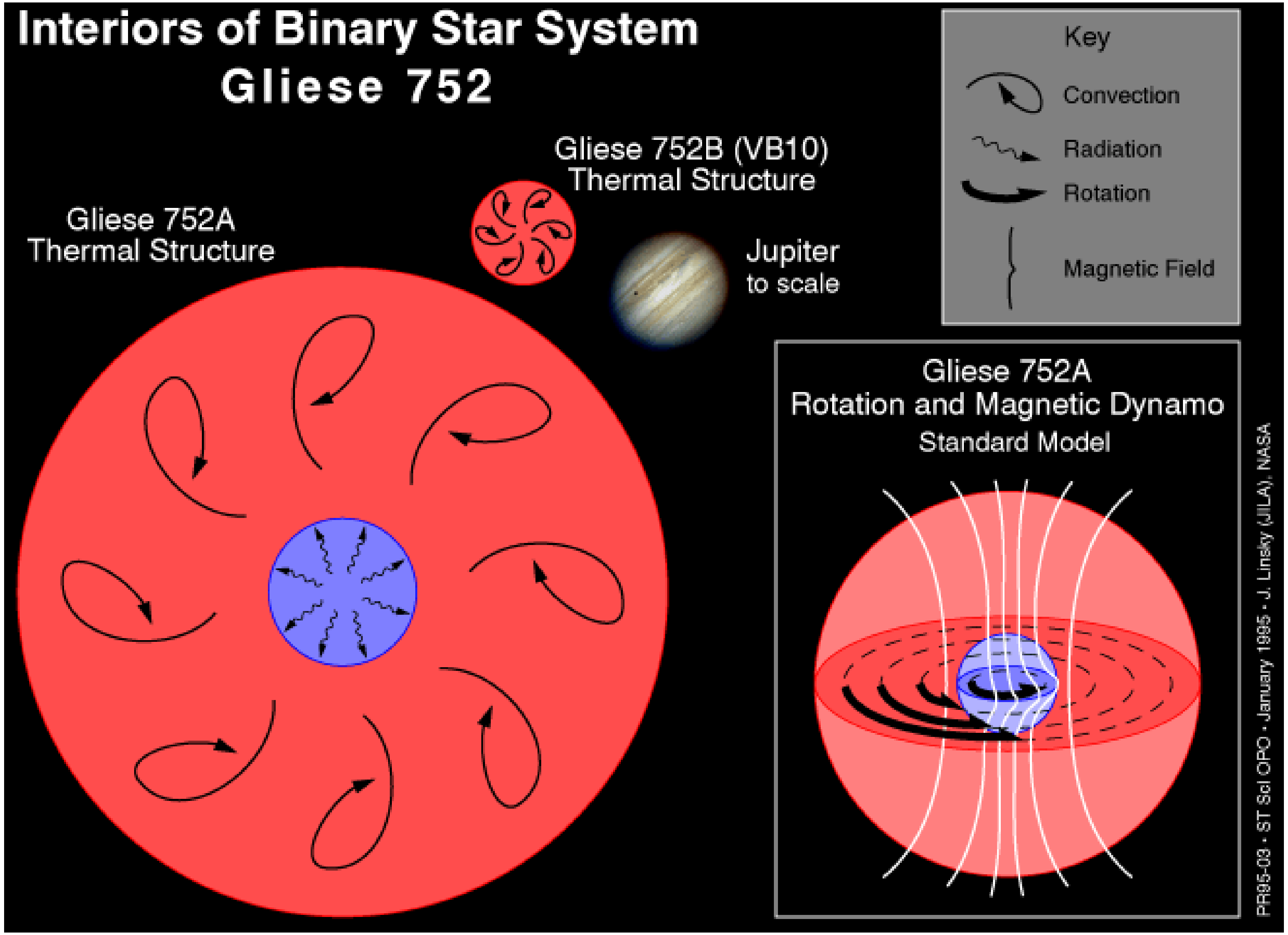


For a **sun-like star**, ionization of H and He, and also the large opacity of H^{-} ions ➔ a convective envelope (outer 30% radius).



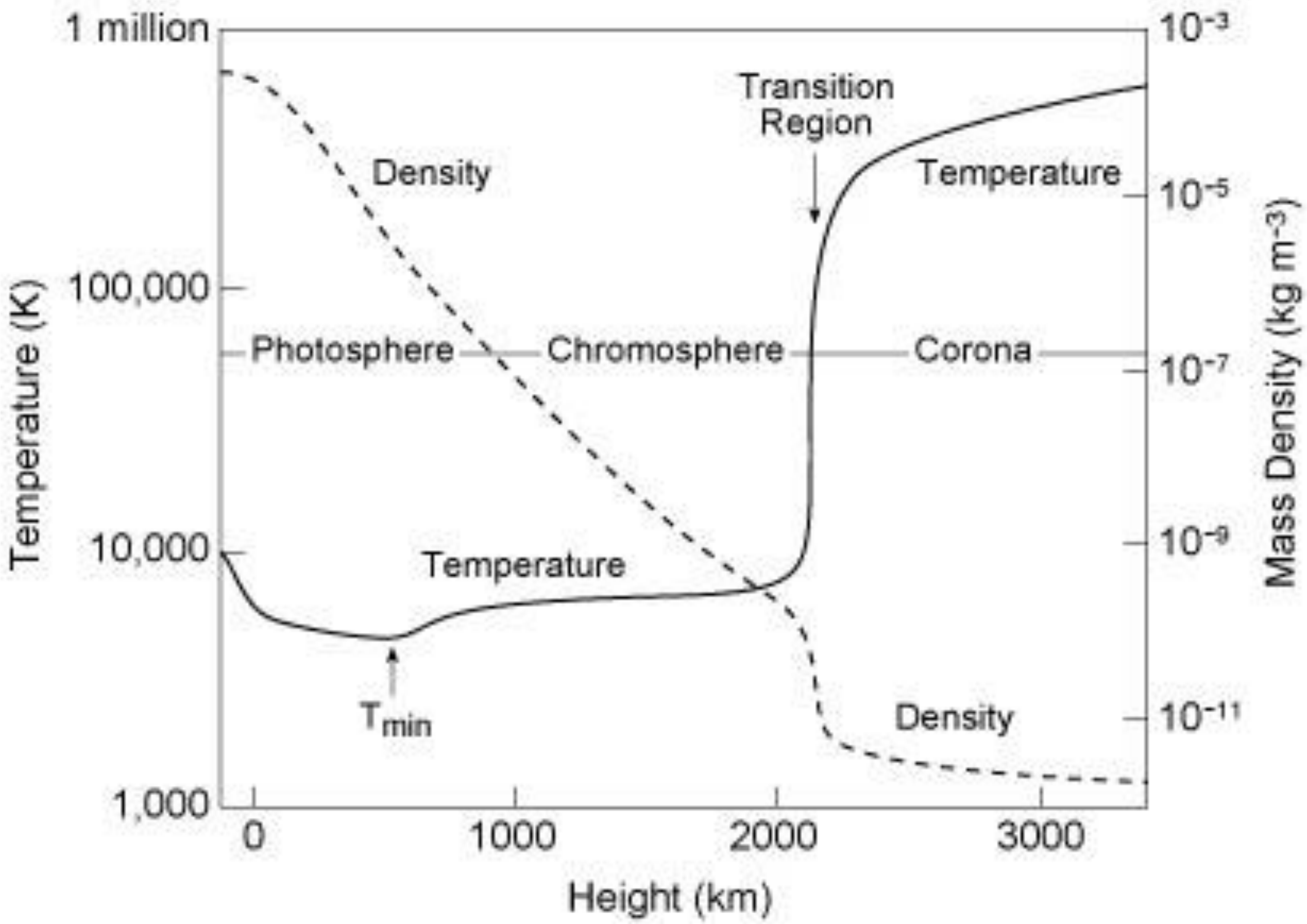
For a **massive star** $\left(M \gtrsim 1.2\ M_{\odot}\right)$, the core produces fierce amount of energy (via CNO) → convective core ➔ a large fraction of material to take part in the thermonuclear reactions



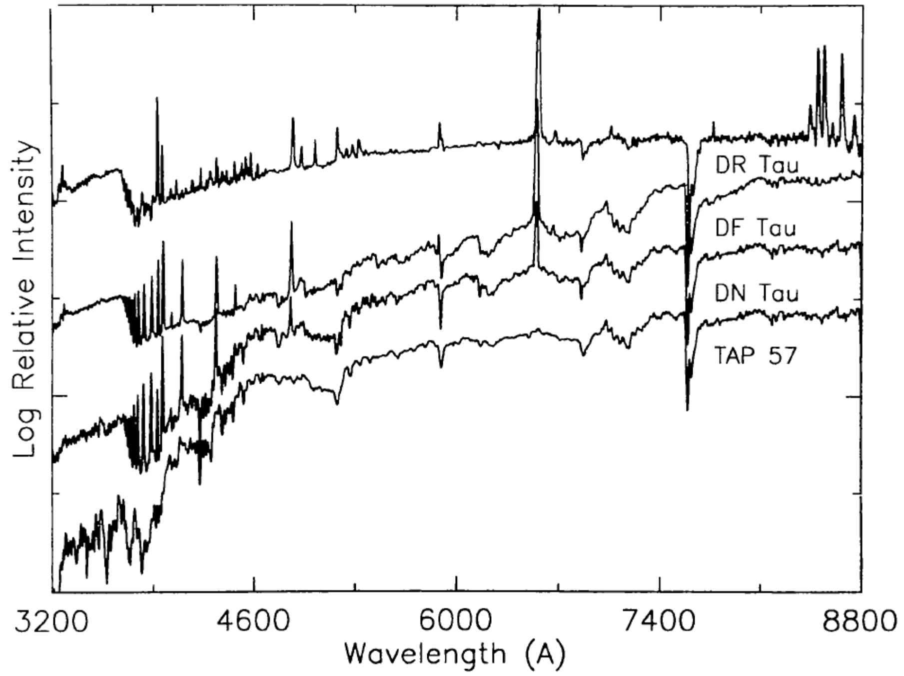


A binary system at 5.74 pc. Gliese 752A (=Wolf 1055) is an M2.5 red dwarf (mass ~0.46 solar, m_V~9.13), whereas Gliese 752B (VB 10) is an M8V (mass ~0.075 solar, m_V~17.30).

Structure of the solar atmosphere



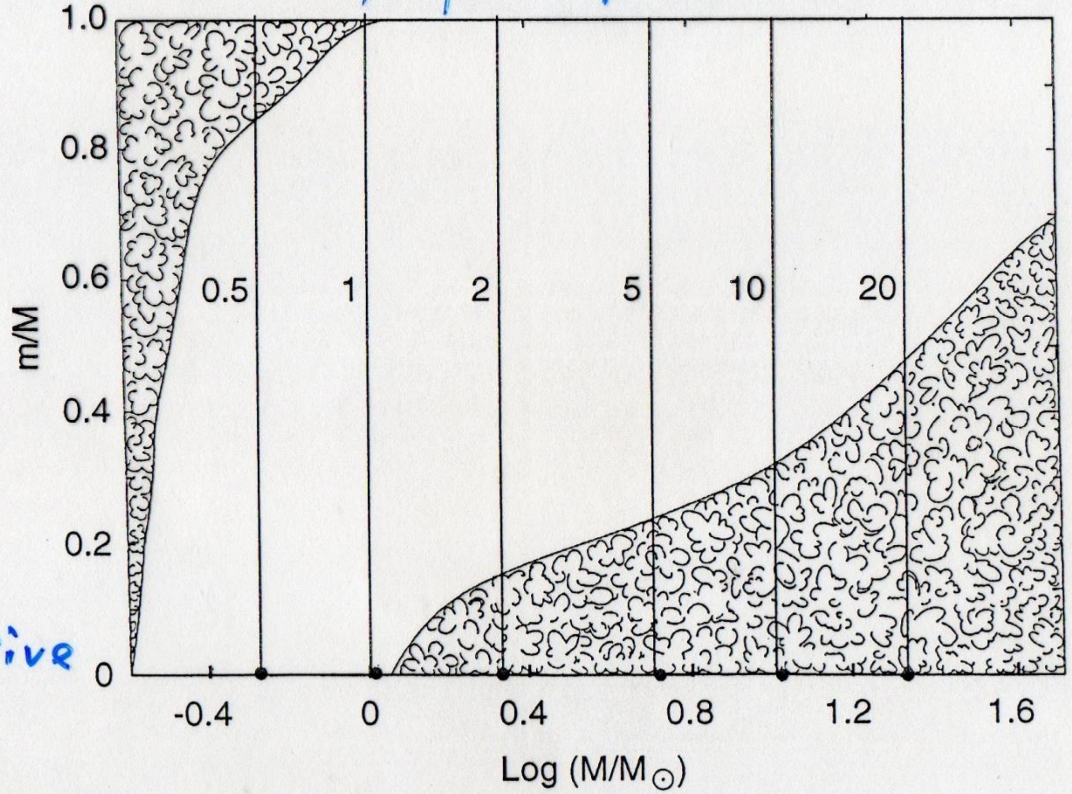
T Tauri stars contracting down to the ZAMS → an enlarged chromosphere → emission spectra



https://ase.tufts.edu/cosmos/view_picture.asp?id=174

M ↑ ⇒ smaller outer convective zone

For the Sun, convective zone only 2% solar mass below the photosphere



$M \lesssim 0.3 M_\odot$

Star fully convective

Figure 8.4 The extent of convective zones (shaded areas) in main-sequence star models as a function of the stellar mass [adapted from R. Kippenhahn & A. Weigert (1990), *Stellar Structure and Evolution*, Springer-Verlag].

Along the ZAMS, $M_* \propto R_*$, so the central density

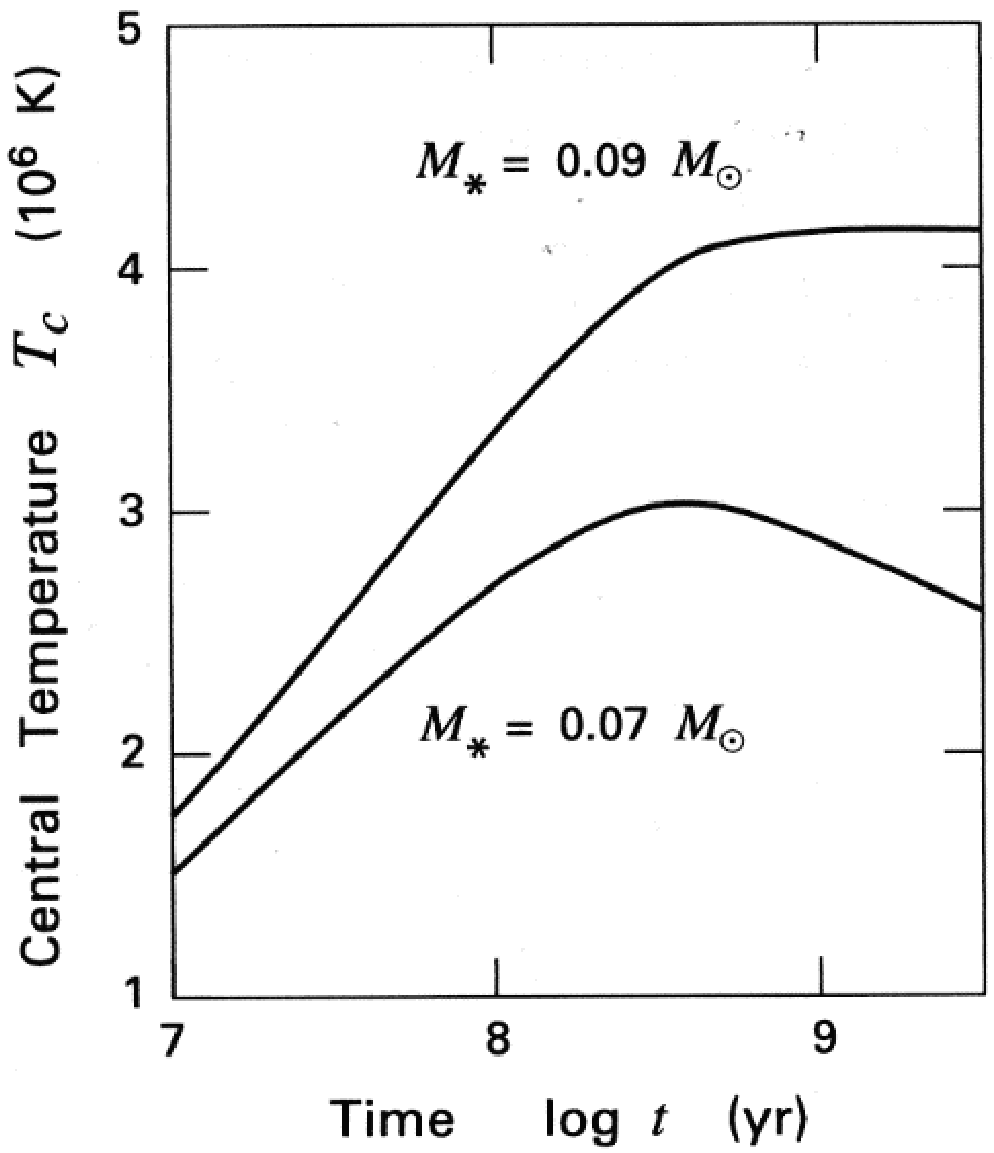
$$\rho_c \propto M_*/R_*^3 \propto M_*^{-2}$$

That is, lower-mass MS stars are denser at the cores

→ to provide sufficient pressure

So temperature may never get high enough for H fusion

→ Degeneracy important



Stahler & Palla, Fig 16.12

Electron Degeneracy

<u>Fermi-Dirac distribution</u> for non-interacting, indistinguishable particles obeying Pauli exclusion principle; applicable to half-integer spin in TE. Examples of fermions include the electron, proton, neutrons, and nuclei with odd mass numbers, e.g., ^{3}He (2 e^{-}, 2 p^{+}, 1 n^{0})

<u>Bose-Einstein distribution</u> for particles not limited to single occupancy of the same energy state. i.e., that do not obey Pauli exclusion principle; with integer values of spin. Example bosons include ^{4}He, the Higgs boson, gauge boson, graviton, meson.

A Fermi gas is called degenerate if the temperature is low in comparison with the Fermi temperature/energy.

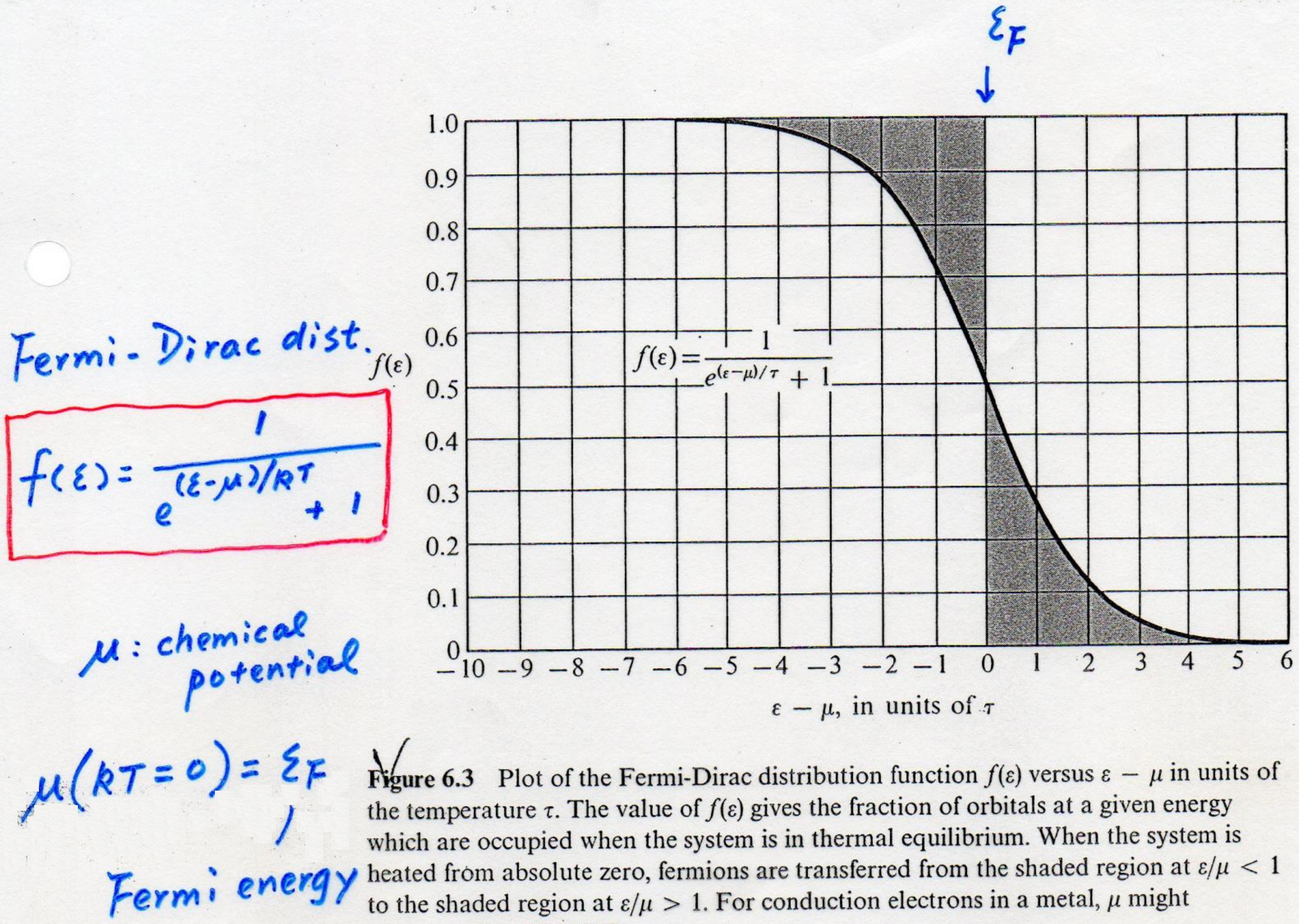


Figure 6.3 Plot of the Fermi-Dirac distribution function $f(\varepsilon)$ versus $\varepsilon - \mu$ in units of the temperature τ. The value of $f(\varepsilon)$ gives the fraction of orbitals at a given energy which are occupied when the system is in thermal equilibrium. When the system is heated from absolute zero, fermions are transferred from the shaded region at $\varepsilon/\mu < 1$ to the shaded region at $\varepsilon/\mu > 1$. For conduction electrons in a metal, μ might correspond to 50 000 K.

Chemical Potential (μ)

- Temperature governs the flow of energy between two systems.
- Chemical potential governs the flow of particles; from higher chemical potential to the lower.

Bose-Einstein dist.

$$f(\varepsilon) = \frac{1}{e^{(\varepsilon-\mu)/kT} - 1}$$

Classical Limit

$$f(\varepsilon) \ll 1$$

$$f(\varepsilon) = e^{-(\varepsilon-\mu)/kT}$$

Maxwell-Boltzmann dist.

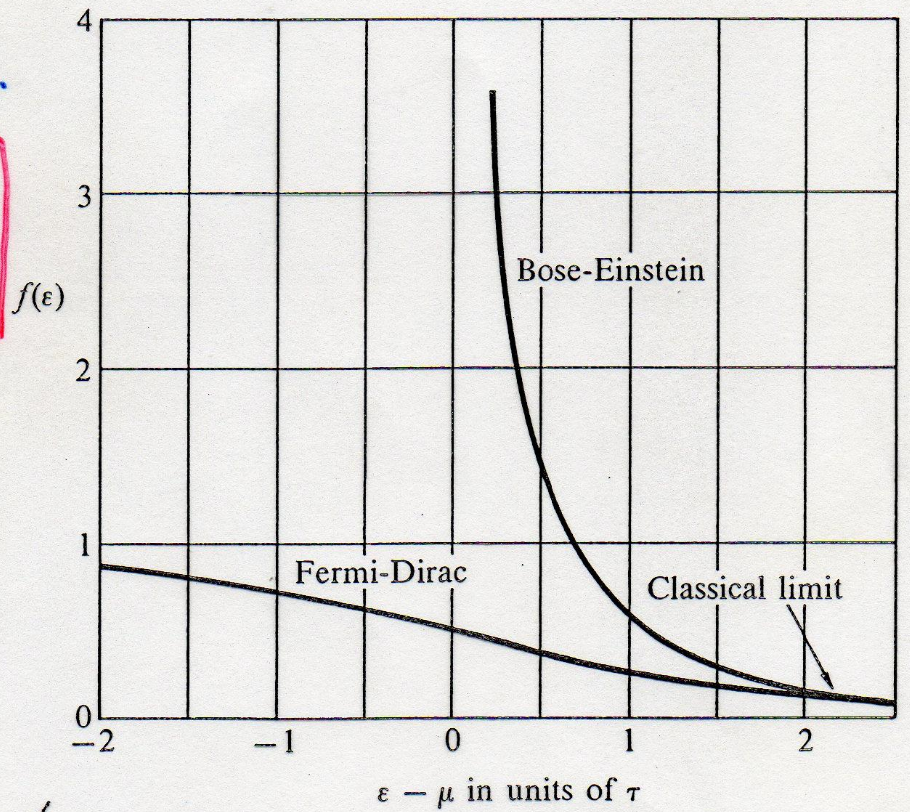


Figure 6.6 Comparison of Bose-Einstein and Fermi-Dirac distribution functions. The classical regime is attained for $(\varepsilon - \mu) \gg \tau$, where the two distributions become nearly identical. We shall see in Chapter 7 that in the degenerate regime at low temperature the chemical potential μ for a FD distribution is positive, and changes to negative at high temperature.

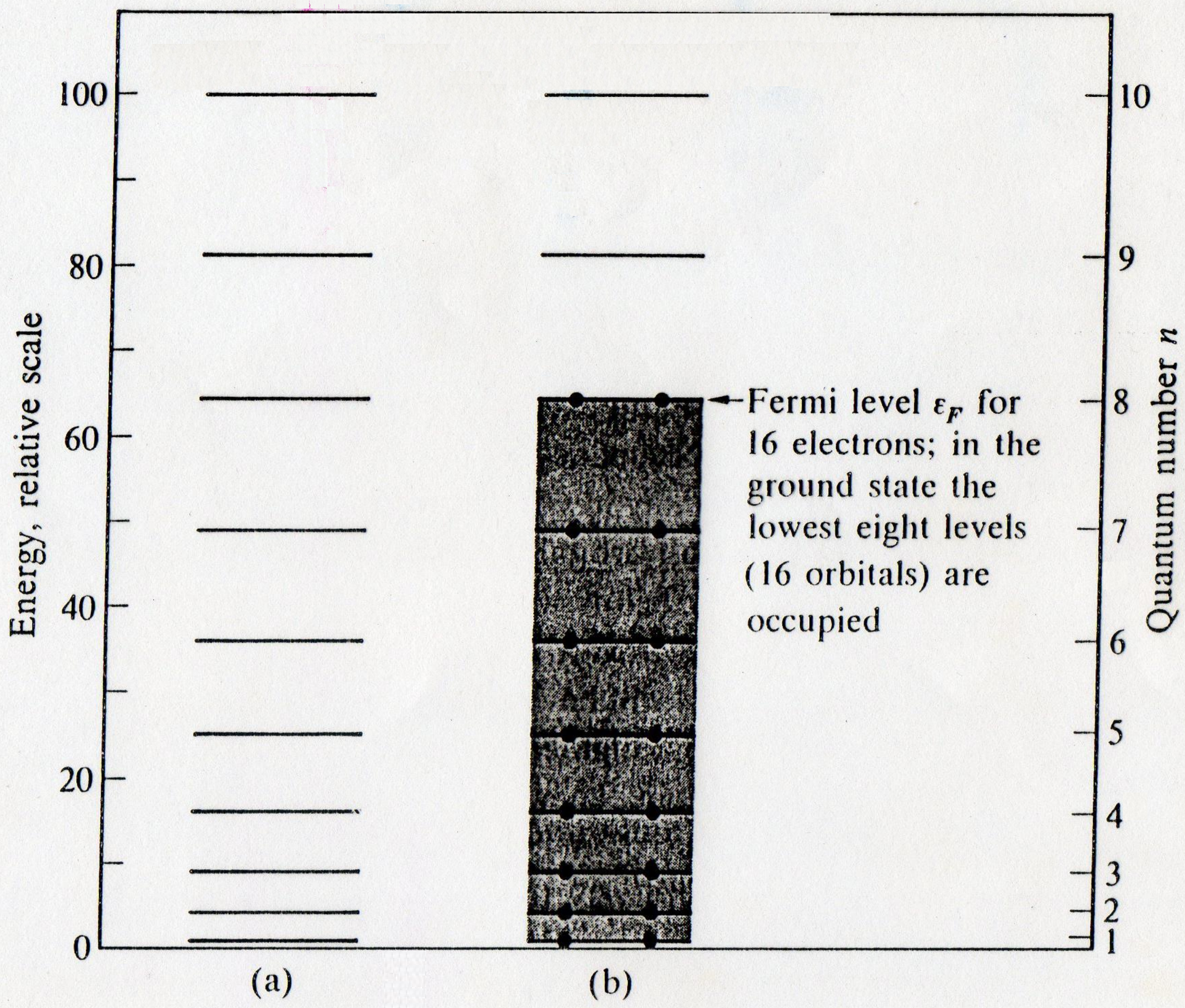


Figure 7.1 (a) The energies of the orbitals $n = 1, 2, \ldots, 10$ for an electron confined to a line of length L. Each level corresponds to two orbitals, one for spin up and one for spin down. (b) The ground state of a system of 16 electrons. Orbitals above the shaded region are vacant in the ground state.

Fermi level

⟶ Fermi energy;

Fermi momentum

Gas Equation of State $P = P(\rho, T)$

In general, the pressure integral (momentum transfer)

$$P = \frac{1}{3}\int_0^\infty vp\, n(p)\, dp$$

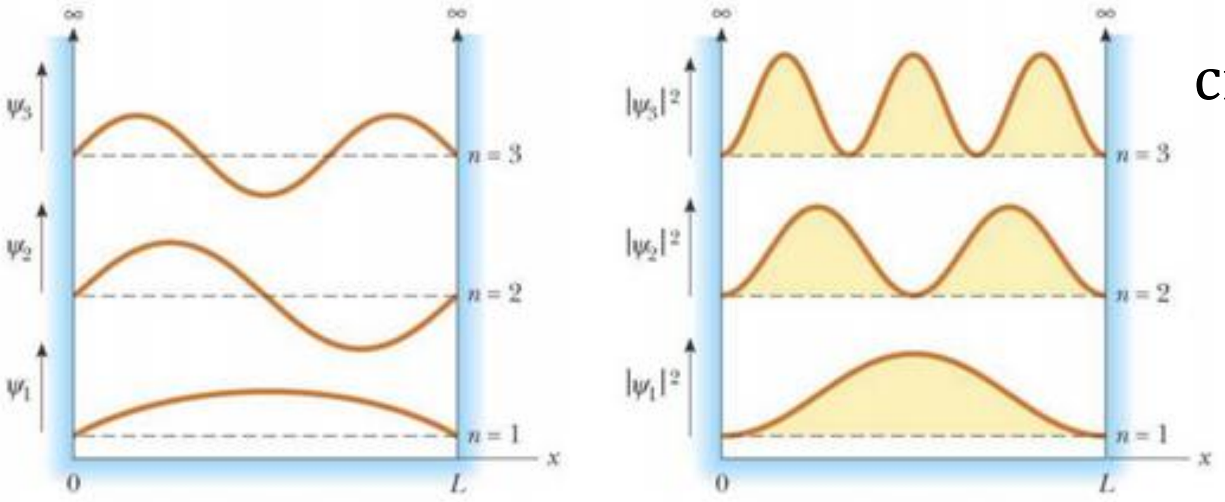
For an deal gas $P \propto \rho T$

For a degenerate electron gas, P independent of T,

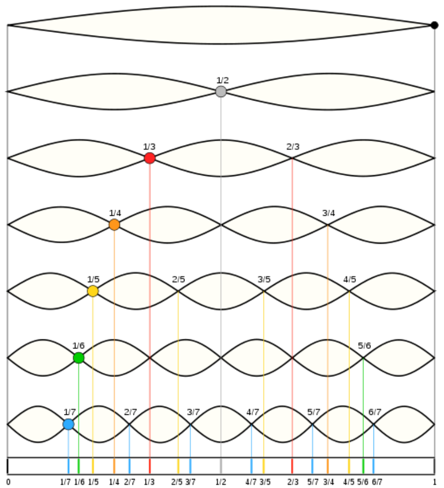
$P \propto \rho^{5/3}$ (non-relativistic)

$P \propto \rho^{4/3}$ (extremely relativistic)

Particle in a Box



cf. standing wave in a string



$\Psi = 0$ at the walls

→ De Broglie wavelength

$$\lambda_n = 2L/n\,, \quad n = 1, 2, 3, \;\ldots$$

$$L = \tfrac{1}{2}\lambda, \tfrac{2}{2}\lambda, \tfrac{3}{2}\lambda, \ldots$$

Since $\lambda_n = {}^{h}/_{p} = {}^{h}/_{mv} \;\rightarrow\; E_K = {}^{1}/_{2}\; mv^2 = (mv)^2/2m = \dfrac{h^2}{2m\lambda^2}$

No potential → $E_n = \; (mv)^2/2m = \dfrac{h^2}{2m\lambda_n^2} = \dfrac{n^2h^2}{8mL^2} = \dfrac{1}{2m}\dfrac{n^2\pi^2\hbar^2}{L^2}$

Within the box, the Schrödinger equation,

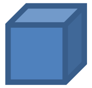
$$\frac{d^2\psi}{dx^2} + \frac{2m}{\hbar^2} E\psi = 0 \rightarrow \psi_n = \sqrt{\frac{2}{L}} \sin\frac{n\pi x}{L}$$

At the <u>center</u>, ψ_1, ψ_3 probability → max

ψ_2 probability $= 0$

c.f. classical physics: <u>same probability everywhere</u> in the box

Consider an atom in a box of volume $V = \ell^3$



wave equation $$-\frac{\hbar^2}{2m}\nabla^2\psi = \varepsilon\,\psi$$

energies, $$\varepsilon_n = \frac{\hbar^2}{2m}\left(\frac{\pi}{\ell}\right)^2\left[n_x^2 + n_y^2 + n_z^2\right]$$

where n_i's are quantum no's

any positive integer

(n_i)

In the phase space

$$\varepsilon_F = \frac{\hbar^2}{2m}\left(\frac{\pi n_F}{\ell}\right)^2$$

n_F: radius that separates filled & empty states

$$\varepsilon_F = \frac{\hbar^2}{2m}\left(\frac{\pi\, n_F}{\ell}\right)^2$$

For N electrons

$$N_e = 2 \times \frac{1}{8} \times \frac{4}{3}\pi n_F^3 \qquad n_F = \left(\frac{3}{\pi} N_e\right)^{1/3}$$

one octant

2 spin states

$$\therefore \quad \varepsilon_F = \frac{\hbar^2}{2m}\,\frac{\pi^2}{V^{2/3}}\left(\frac{3}{\pi}N_e\right)^{2/3} = \frac{\hbar^2}{2m}\left(3\pi^2\frac{N_e}{V}\right)^{2/3}$$

$$\boxed{\varepsilon_F = \frac{\hbar^2}{2m}\left(3\pi^2 n_e\right)^{2/3}} \sim n_e^{2/3}$$

electron concentration

Fermi energy: the <u>highest</u> energy level filled at temperature zero

Fermi energy of degenerate fermion gases

Phase of matter	Particles	E_F	$T_F = \varepsilon_F / k_B [\,\mathrm{K}]$
Liquid ^3He	atoms	4×10^{-4}eV	4.9
Metal	electrons	2–10 eV	5×10^4
White dwarfs	electrons	0.3 MeV	3×10^9
Nuclear matter	nucleons	30 MeV	3×10^{11}
Neutron stars	neutrons	300 MeV	3×10^{12}

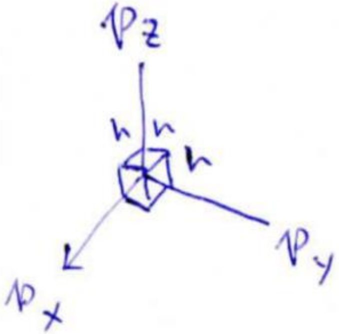
$$\varepsilon_F = \frac{\hbar^2}{2m_e} (3\pi^2 n_e)^{2/3}$$

Considering the problem in terms of **momentum**.

Degenerate State

$$E_n = \frac{\hbar^2}{2m}\left(\frac{n\pi}{\ell}\right)^2 \Rightarrow E_f = \frac{\hbar^2}{2m}\left(\frac{n_F\pi}{\ell}\right)^2 = \frac{\hbar^2}{2m}\left(3\pi^2 n_e\right)^{2/3}$$

Total $N_e = 2\cdot\frac{1}{8}\cdot\frac{4}{3}\pi n_F^3 = \frac{\pi}{3}n_F^3 \Rightarrow n_F = \left(\frac{3}{\pi}n_e\right)^{1/3}$



Uncertainty Principle $\Delta V \, \Delta^3 p \lesssim h^3$

$$2\cdot 4\pi p^2 \, dp = h^3 \cdot n_e(p)\, dp$$

Upto p_F, $2\cdot\frac{4}{3}\pi p_F^3 = N_e = n_e\cdot h^3 \Rightarrow p_F = \left(\frac{3h^3}{8\pi}n_e\right)^{1/3}$

Pressure and Momentum

$$\boldsymbol{P} = \frac{1}{3}\int_0^\infty v\, p\, n(p) dp$$

Non-relativistic

Pressure Integral $P = \frac{1}{3}\int_0^\infty n(p)\, v\, p\, dp$ (use $v = p/m_e$)

$$= \frac{1}{3}\int_0^{p_F} \frac{8\pi p^2}{h^3}\frac{p}{m_e}\, p\, dp$$

$$= \frac{8\pi}{3 m_e h^3}\frac{1}{5} p_F^5 = \frac{8\pi}{15 m_e h^3} p_F^5$$

For electrons, $n_e = \frac{\rho}{\mu_e m_H}$ $\therefore P = \frac{h^2}{20 m_e}\left(\frac{3}{\pi}\right)^{2/3}\left(\frac{\rho}{\mu_e m_H}\right)^{5/3}$

In the non-relativistic case

$$P_{\rm e,deg}^{\rm NR} = \frac{h^2}{20m_e}\left(\frac{3}{\pi}\right)^{2/3}\frac{1}{m_{\rm H}^{5/3}}\left(\frac{\rho}{\mu_e}\right)^{5/3}$$

$$= 1.00\times10^{13}\left(\frac{\rho}{\mu_e}\right)^{5/3} \quad [\rm cgs]$$

$$\propto \rho^{5/3}$$

In the extremely relativistic case $v \rightarrow c$ in the pressure integral

$$P_{\rm e,deg}^{\rm ER} = \frac{hc}{8}\left(\frac{3}{\pi}\right)^{1/3}\frac{1}{m_{\rm H}^{3/4}}\left(\frac{\rho}{\mu_e}\right)^{4/3}$$

$$= 1.24\times10^{15}\left(\frac{\rho}{\mu_e}\right)^{4/3} \quad [\rm cgs]$$

$$\propto \rho^{4/3}$$

For a composition devoid of hydrogen, and not very rich in extremely heavy elements, $\mu_e \approx 2$.

Mass-radius relation for a degenerate electron gas $\quad P \sim \frac{M^2}{R^4}$

In the NR case, $P \propto \rho^{5/3} \sim \left(\frac{M}{R^3}\right)^{5/3} = \frac{M^{5/3}}{R^5} \Longrightarrow \boxed{MR^3 = \text{const}}$

So $M \nearrow$, $R \searrow$, $\rho \nearrow\nearrow$, electrons move ever faster.

$$\log\left(\frac{R}{R_\odot}\right) = -\frac{1}{3}\log\left(\frac{M}{\mathrm{M}_\odot}\right) - \frac{5}{3}\log\mu_e - 1.397$$

In the ER case, $P \propto \rho^{4/3} = \frac{M^{4/3}}{R^4}$, no solution between M and R.

A mass limit for a degenerate electron body (white dwarf)

Chandrasekhar limit $\quad M_{WD} \lesssim 5.8\, M_\odot/\mu_e^2$

$$L = \sigma T_e^4 (4\pi R^2)$$

$$\log\left(\frac{L}{L_\odot}\right) = 4 \log\left(\frac{T_e}{T_{e\odot}}\right) + 2 \log\left(\frac{R}{R_\odot}\right)$$

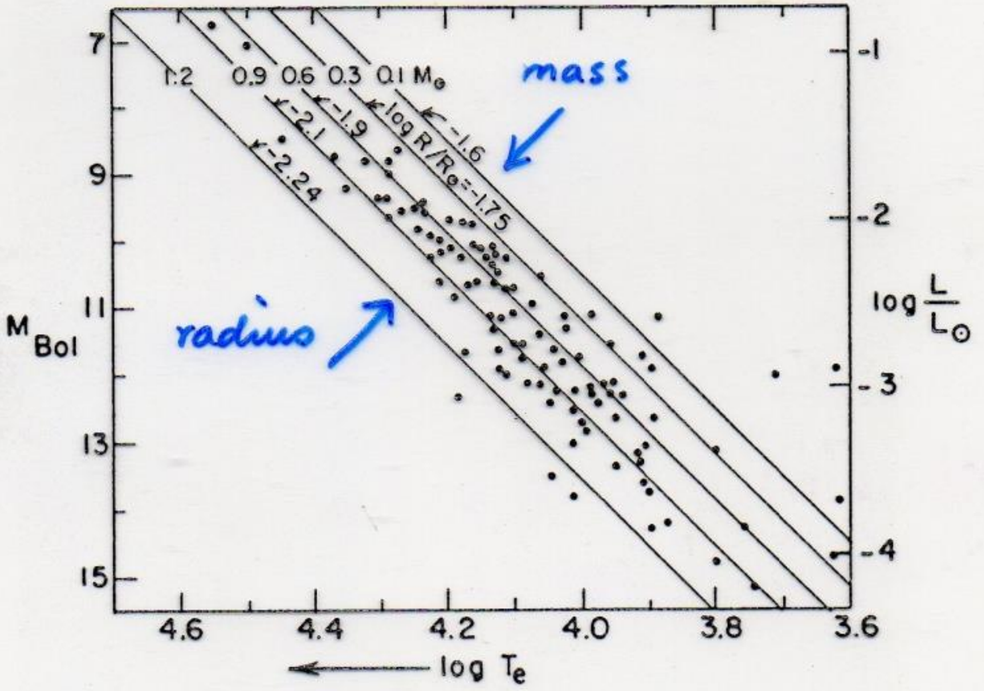
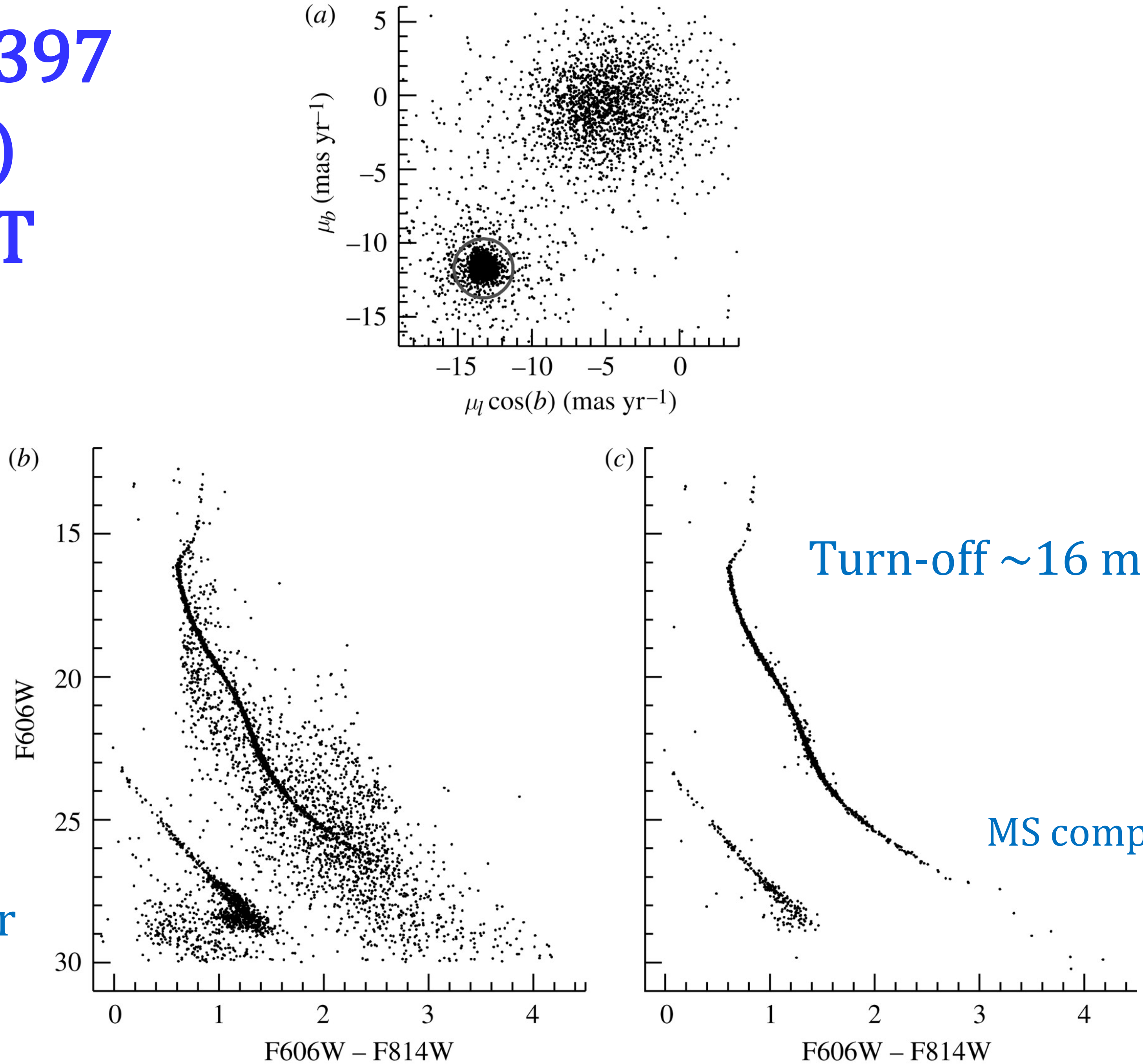


FIGURE 8.14. White dwarf Hertzsprung–Russell diagram. Lines of constant radius are shown. Also shown are the masses based on completely degenerate core models containing elements having $\mu_e = 2$ (after Weidemann (We68)). Reprinted with permission from *Annual Review of Astronomy and Astrophysics*, Vol. 6, ©1968 by Annual Reviews, Inc.).

GC NGC 6397 (~12 Gyr) by the HST

Kalirai 2010



Turn-off ~16 mag, 0.8 $M_\odot$

MS complete ~26 mag, 0.092 $M_\odot$

Detection limit
~30 mag, 0.083 $M_\odot$

WD cooling:
11.47 ± 0.47 Gyr