Stellar Equations of State

Photon "gas"

$$P_R = \frac{2}{c}\int I \cos^2\theta \, \mathrm{d}\omega = aT^4/3 \text{ (for isotropic radiation)}$$

Gas Equation of State $P = P(\rho, T)$

In general, the pressure integral (momentum transfer)

$$P = \frac{1}{3}\int_0^{\infty} v(p)\, p\, n(p)\, dp$$

… derivation in the momentum space

Ideal gas

$$P = n\,kT = \frac{\rho}{\mu\, m_H} kT$$

Electron Degeneracy

…when temperature is exceedingly low
or matter is highly compressed

Compact Objects

Nuclear energy $4\overset{m}{H} - \overset{m}{H_e} = 0.029\, m_H$

mass deficit $= 7 \times 10^{-3}$ g/g

$\therefore$ Energy available $= mc^2 = \underline{6 \times 10^{18}\ \text{erg g}^{-1}}$

Chemical energy $\lesssim 100$ Kcal $\Rightarrow \underline{4 \times 10^{12}\ \text{erg g}^{-1}}$

Gravitational energy e.g. for $\odot$, $\frac{3}{5}\frac{M_\odot^2 G}{R_\odot} \sim 2 \times 10^{48}$ erg

$\Rightarrow \underline{10^{15}\ \text{erg g}^{-1}}$

Accretion $\frac{MG}{r} \cdot \dot{m}$

In general $\frac{\mathcal{E}_{nuc}}{mass} \sim 0.01 c^2$ $\frac{\mathcal{E}_{grav}}{mass} \sim \frac{3GM}{5R}$.

$\uparrow\uparrow$ as $R \downarrow\downarrow$

For very compact objects, large amounts of gravitational energy can be released, perhaps even more than nuclear energy,

$$R \lesssim \frac{MG}{0.01 c^2} \sim 10^7 \text{ cm} \sim 100 \text{ km}, \text{ for } 1 M_\odot$$

cf. Schwarzschild radius $R_S \equiv \frac{2GM}{c^2} \sim 3$ km, for $1 M_\odot$

Atoms in a white dwarf are fully ionized and the e^- gas is degenerate.

1844 Bessel observed the oscillated path of Sirius

1862 Sirius B discoved by Clark

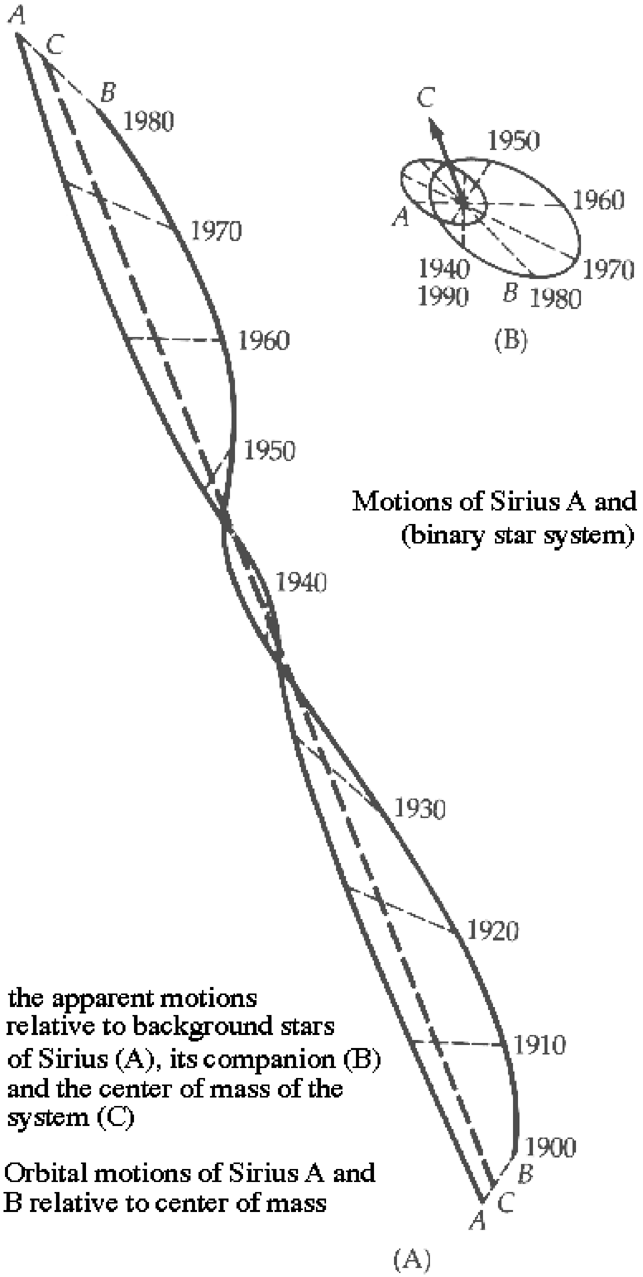
$M(\text{Sirius B}) \sim 2 \times 10^{33}$ g ← orbit

$R(\text{Sirius B}) \sim 2 \times 10^{9}$ cm ← surface temp. and radiation

cf $R_\odot \sim 7 \times 10^{10}$ cm

$$\bar{\rho}_{\text{Sirius B}} = \frac{M}{\frac{4}{3}\pi R^3} \sim 0.7 \times 10^{5} \text{ g cm}^{-3}$$

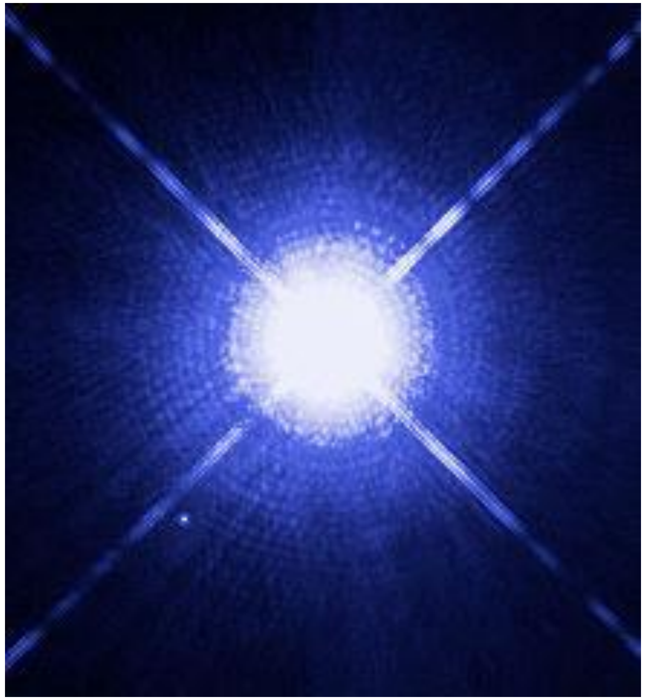
cf $\bar{\rho}_{\text{sun}} \sim 1 \text{ g cm}^{-3}$



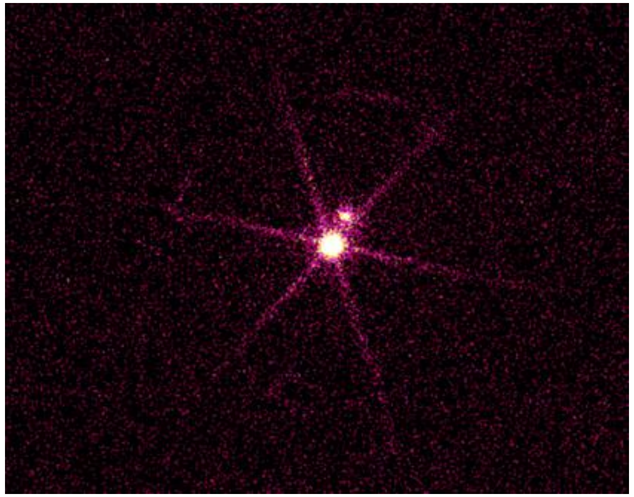
Motions of Sirius A and B
(binary star system)

(A) the apparent motions relative to background stars of Sirius (A), its companion (B) and the center of mass of the system (C)

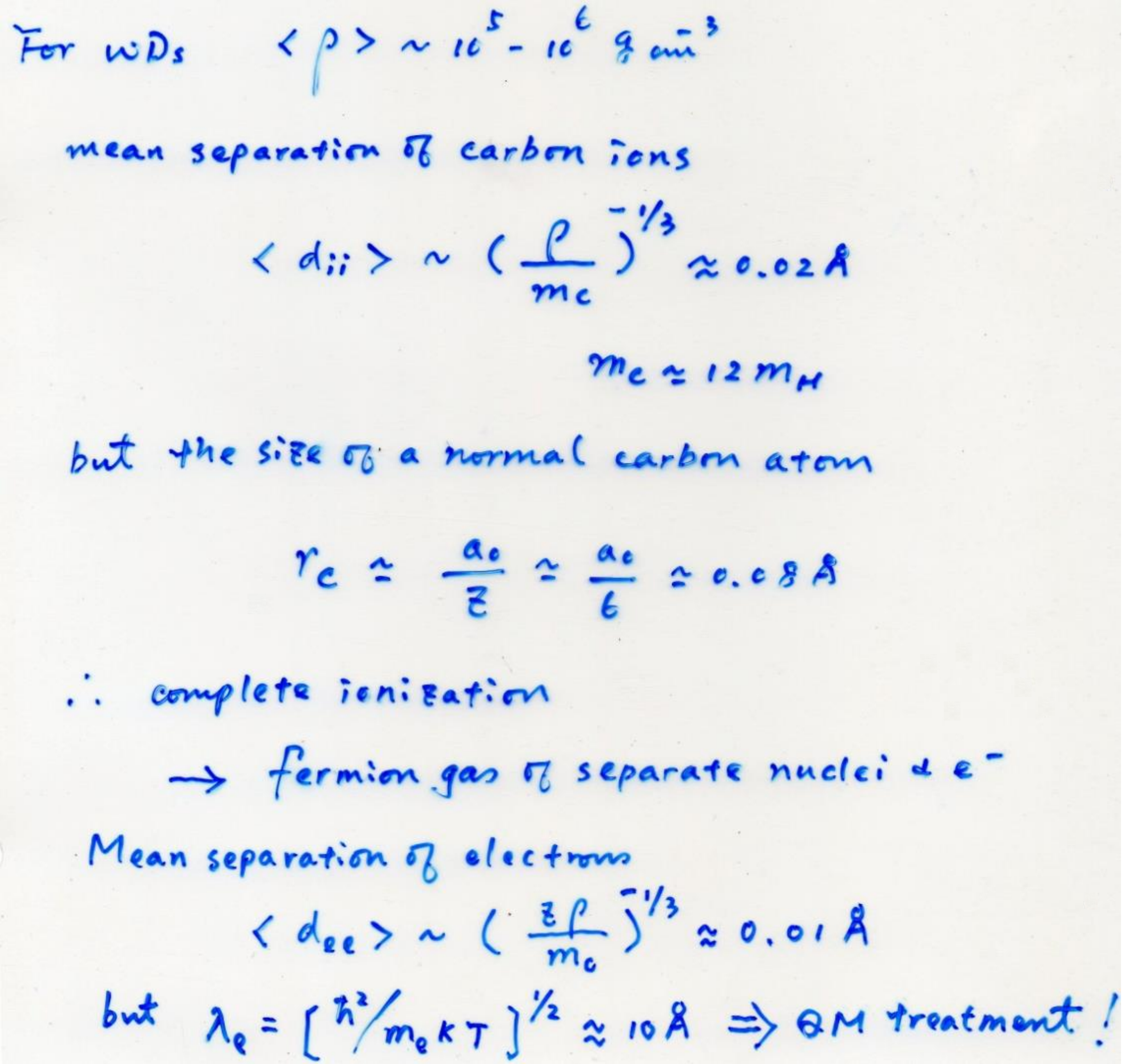
(B) Orbital motions of Sirius A and B relative to center of mass



Sirius A and B by the *HST*



Sirius B and A by the *Chandra Observatory*



For WDs $\langle \rho \rangle \sim 10^5 - 10^6 \text{ g cm}^{-3}$

mean separation of carbon ions

$$\langle d_{ii} \rangle \sim \left(\frac{\rho}{m_c} \right)^{-1/3} \approx 0.02 \text{ Å}$$

$$m_c \simeq 12 m_H$$

but the size of a normal carbon atom

$$r_c \simeq \frac{a_0}{Z} \simeq \frac{a_0}{6} \simeq 0.08 \text{ Å}$$

∴ complete ionization

→ fermion gas of separate nuclei & e^-

Mean separation of electrons

$$\langle d_{ee} \rangle \sim \left(\frac{Z\rho}{m_c} \right)^{-1/3} \approx 0.01 \text{ Å}$$

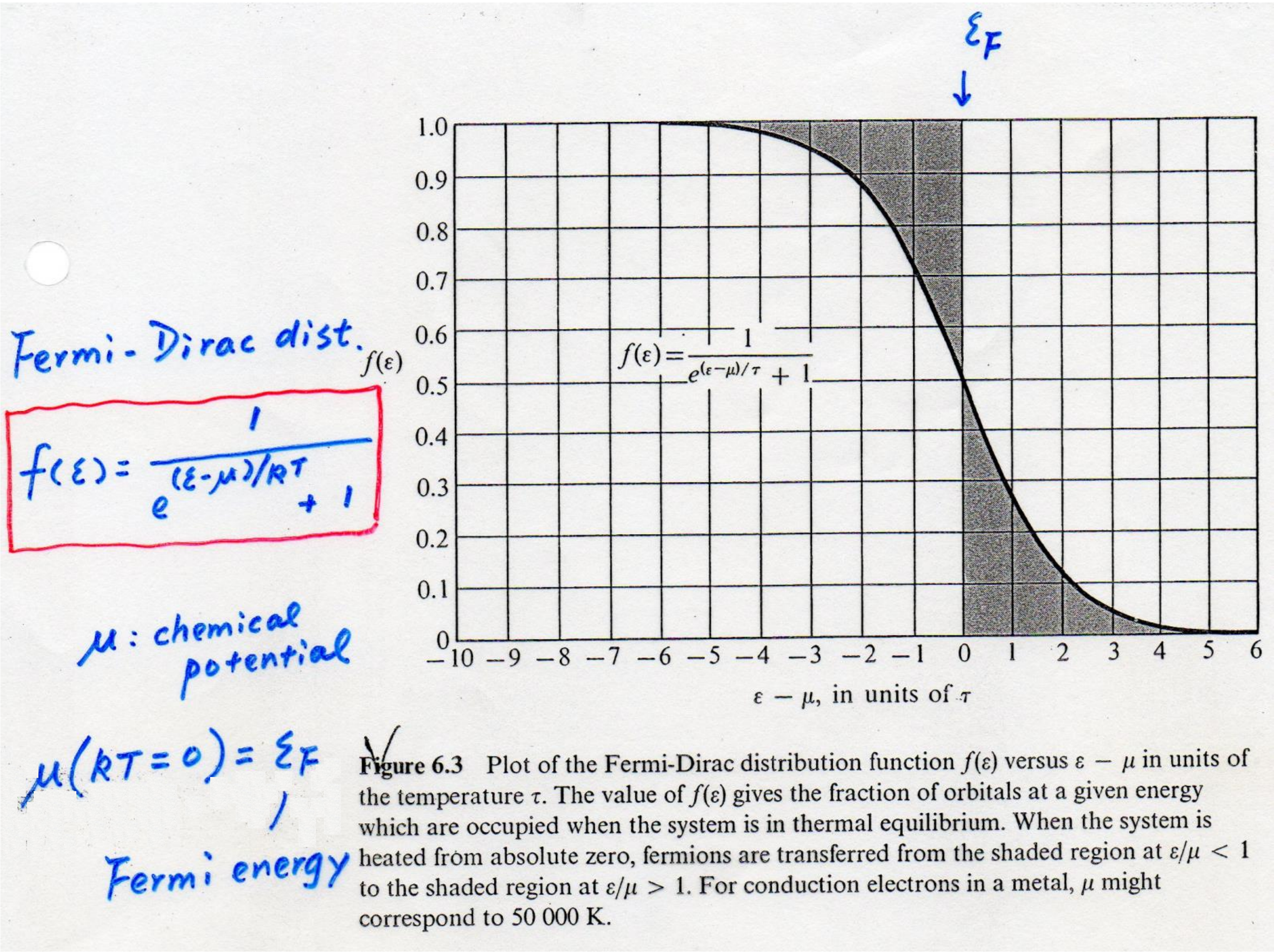
but $\lambda_e = \left[\hbar^2 / m_e k T \right]^{1/2} \approx 10 \text{ Å} \Rightarrow$ QM treatment!

electron gas

Fermi-Dirac distribution for non-interacting, indistinguishable particles obeying Pauli exclusion principle; applicable to half-integer spin in TE. Examples of fermions include the electron, proton, neutrons, and nuclei with odd mass numbers, e.g., ^{3}He (2 e^-, 2 p^+, 1 n^0)

Bose-Einstein distribution for particles not limited to single occupancy of the same energy state. i.e., that do not obey Pauli exclusion principle; with integer values of spin. Example bosons include ^{4}He, the Higgs boson, gauge boson, graviton, meson.

A Fermi gas is called degenerate if the temperature is low in comparison with the Fermi temperature/energy.



Fermi-Dirac dist.

$$\boxed{f(\varepsilon) = \frac{1}{e^{(\varepsilon-\mu)/kT} + 1}}$$

μ : chemical potential

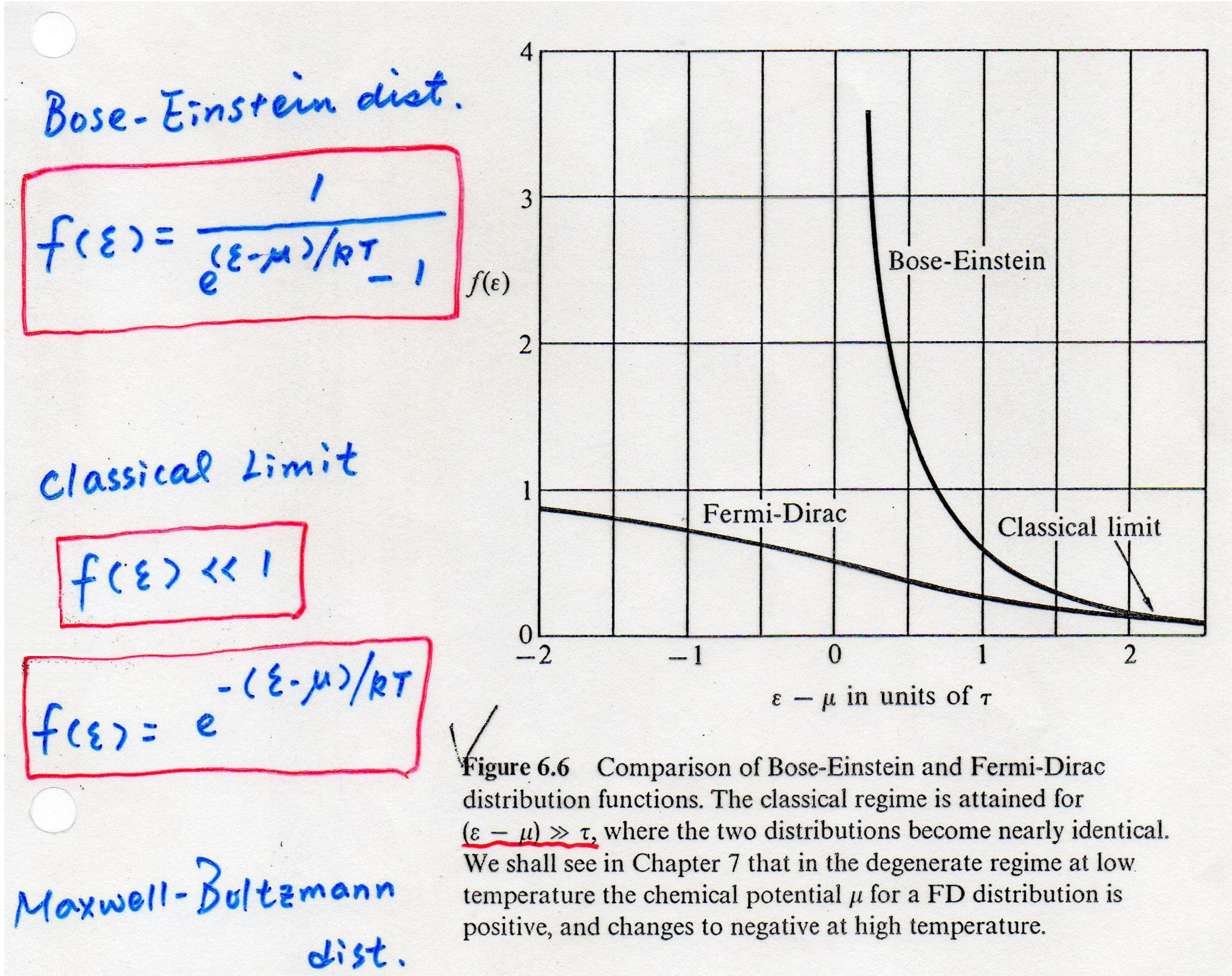
$\mu(kT=0) = \varepsilon_F$

Fermi energy

✓ **Figure 6.3** Plot of the Fermi-Dirac distribution function $f(\varepsilon)$ versus $\varepsilon - \mu$ in units of the temperature τ. The value of $f(\varepsilon)$ gives the fraction of orbitals at a given energy which are occupied when the system is in thermal equilibrium. When the system is heated from absolute zero, fermions are transferred from the shaded region at $\varepsilon/\mu < 1$ to the shaded region at $\varepsilon/\mu > 1$. For conduction electrons in a metal, μ might correspond to 50 000 K.

Chemical Potential (μ)

- Temperature governs the flow of energy between two systems.
- Chemical potential governs the flow of particles; from higher chemical potential to the lower.



Bose-Einstein dist.

$$\boxed{f(\varepsilon) = \frac{1}{e^{(\varepsilon-\mu)/kT} - 1}}$$

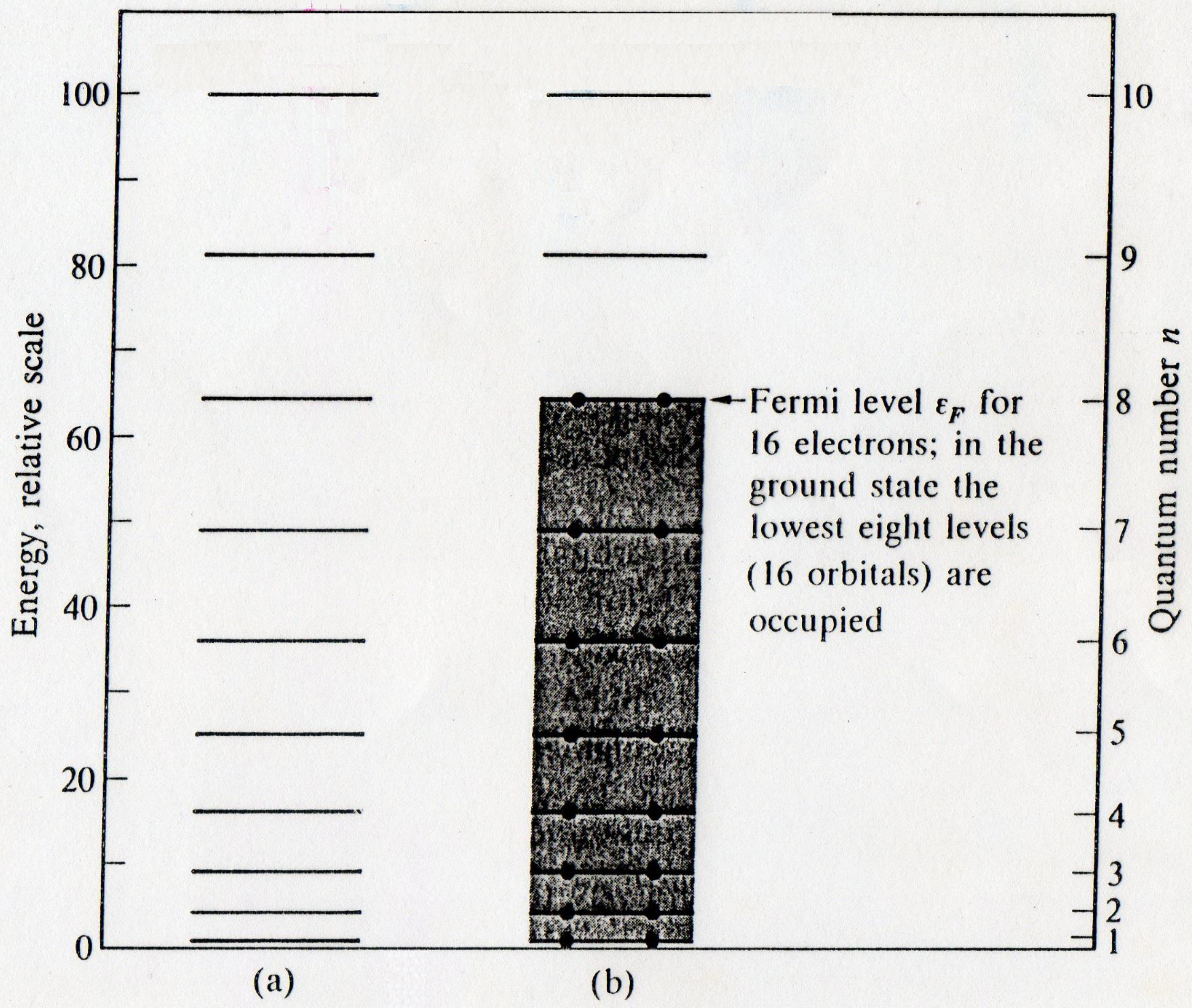
Classical Limit

$$\boxed{f(\varepsilon) \ll 1}$$

$$\boxed{f(\varepsilon) = e^{-(\varepsilon-\mu)/kT}}$$

Maxwell-Boltzmann dist.

✓ **Figure 6.6** Comparison of Bose-Einstein and Fermi-Dirac distribution functions. The classical regime is attained for $\underline{(\varepsilon - \mu) \gg \tau}$, where the two distributions become nearly identical. We shall see in Chapter 7 that in the degenerate regime at low temperature the chemical potential μ for a FD distribution is positive, and changes to negative at high temperature.



Fermi level ε_F for 16 electrons; in the ground state the lowest eight levels (16 orbitals) are occupied

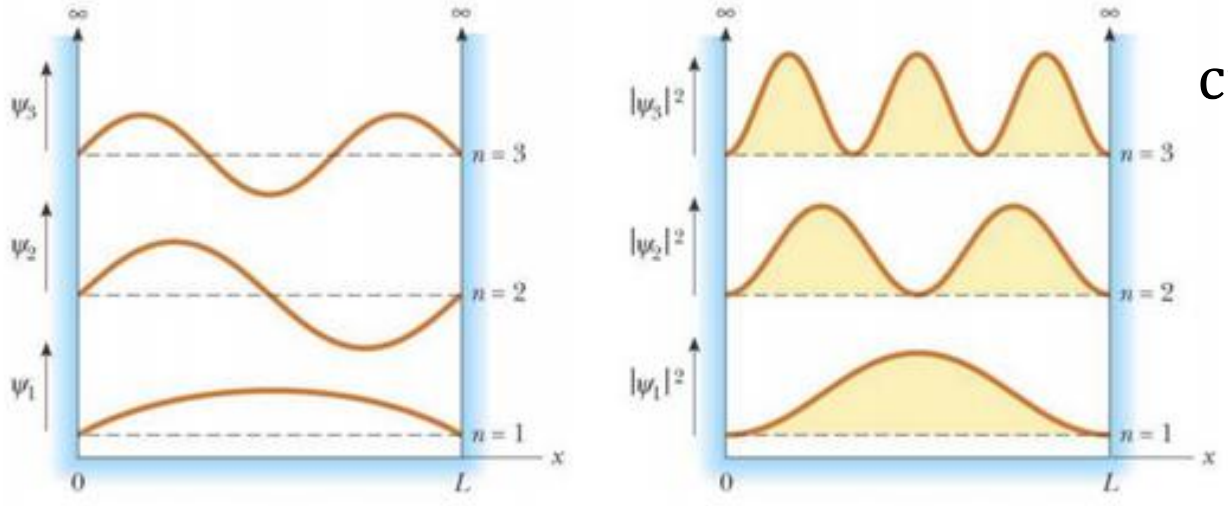
Figure 7.1 (a) The energies of the orbitals $n = 1, 2, \ldots, 10$ for an electron confined to a line of length L. Each level corresponds to two orbitals, one for spin up and one for spin down. (b) The ground state of a system of 16 electrons. Orbitals above the shaded region are vacant in the ground state.

Fermi level

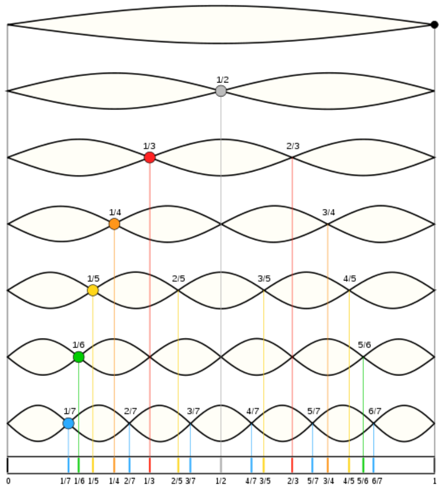
⟶ Fermi energy;

Fermi momentum

Particle in a Box



cf. standing wave in a string



$\Psi = 0$ at the walls

→ De Broglie wavelength

$$\lambda_n = 2L/n\,, \quad n = 1, 2, 3, \ \ldots$$

$$L = \tfrac{1}{2}\lambda, \tfrac{2}{2}\lambda, \tfrac{3}{2}\lambda, \ldots$$

Since $\lambda_n = {}^{h}/_{p} = {}^{h}/_{mv} \rightarrow E_K = {}^{1}/_{2}\ mv^2 = (mv)^2/2m = \dfrac{h^2}{2m\lambda^2}$

No potential → $E_n = \ (mv)^2/2m = \dfrac{h^2}{2m\lambda_n^2} = \dfrac{n^2h^2}{8mL^2} = \dfrac{1}{2m}\dfrac{n^2\pi^2\hbar^2}{L^2}$

Within the box, the Schrödinger equation,

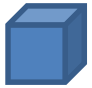
$$\frac{d^2\psi}{dx^2} + \frac{2m}{\hbar^2} E\psi = 0 \rightarrow \psi_n = \sqrt{\frac{2}{L}} \sin\frac{n\pi x}{L}$$

At the <u>center</u>, ψ_1, ψ_3 probability → max

ψ_2 probability $= 0$

c.f. classical physics: <u>same probability everywhere</u> in the box

Consider an atom in a box of volume $V = \ell^3$



wave equation $$-\frac{\hbar^2}{2m}\nabla^2\psi = \varepsilon\,\psi$$

energies, $$\varepsilon_n = \frac{\hbar^2}{2m}\left(\frac{\pi}{\ell}\right)^2\left[n_x^2 + n_y^2 + n_z^2\right]$$

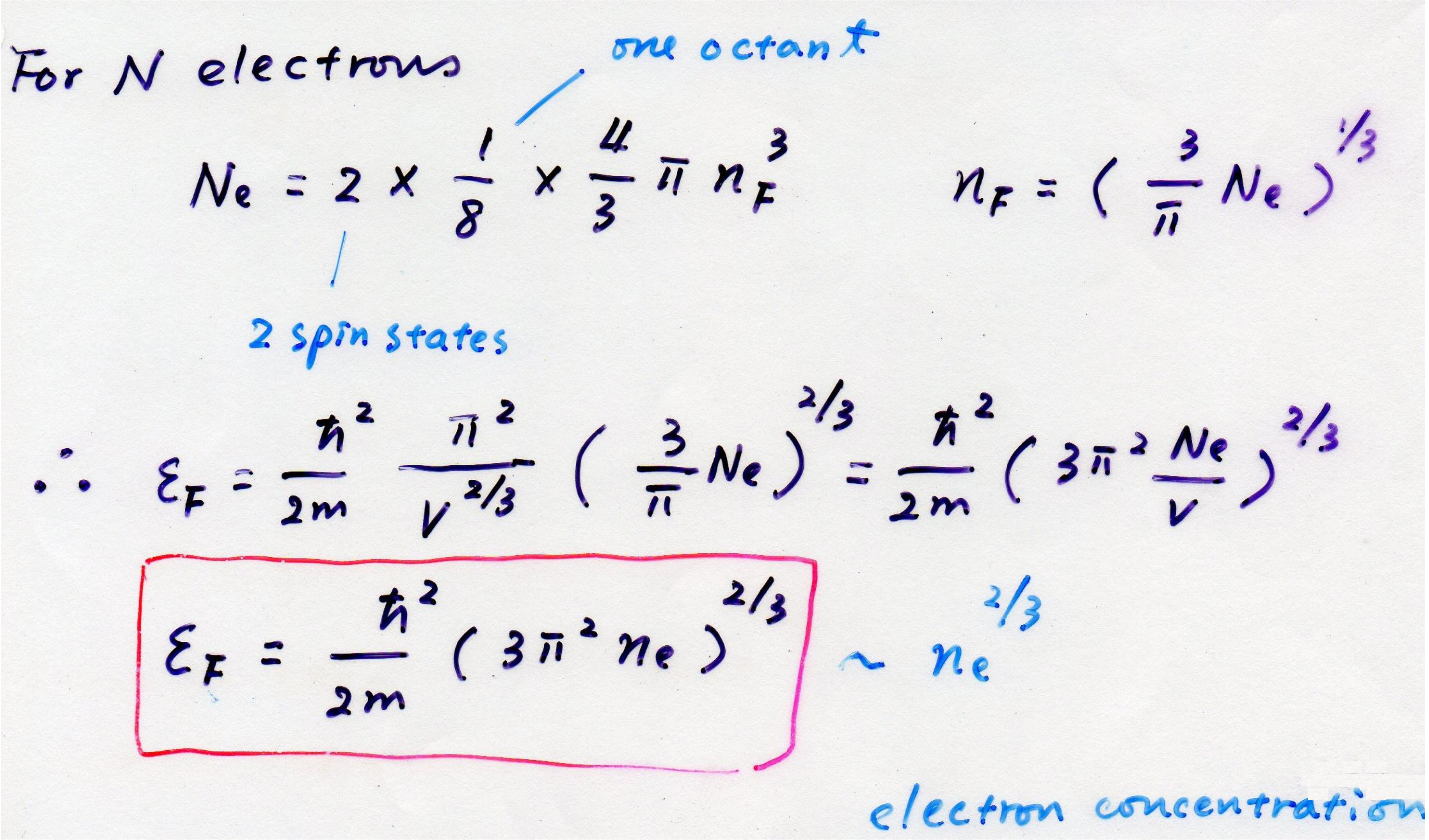
where n_i's are quantum no's
any positive integer

(n_i)
In the phase space

$$\varepsilon_F = \frac{\hbar^2}{2m}\left(\frac{\pi n_F}{\ell}\right)^2$$

n_F: radius that separates
filled & empty states

$$\varepsilon_F = \frac{\hbar^2}{2m}\left(\frac{\pi\, n_F}{\ell}\right)^2$$



For N electrons

one octant

$$N_e = 2 \times \frac{1}{8} \times \frac{4}{3}\pi n_F^3 \qquad n_F = \left(\frac{3}{\pi} N_e\right)^{1/3}$$

2 spin states

$$\therefore\ \varepsilon_F = \frac{\hbar^2}{2m}\,\frac{\pi^2}{V^{2/3}}\left(\frac{3}{\pi}N_e\right)^{2/3} = \frac{\hbar^2}{2m}\left(3\pi^2\frac{N_e}{V}\right)^{2/3}$$

$$\boxed{\varepsilon_F = \frac{\hbar^2}{2m}\left(3\pi^2 n_e\right)^{2/3}} \sim n_e^{2/3}$$

electron concentration

Fermi energy: the <u>highest</u> energy level filled at temperature zero

Fermi energy of degenerate fermion gases

Phase of matter	Particles	E_F	$T_F = \mathcal{E}_F/k_B$[K]
Liquid ^{3}He	atoms	4×10^{-4}eV	4.9
Metal	electrons	2–10 eV	5×10^4
White dwarfs	electrons	0.3 MeV	3×10^9
Nuclear matter	nucleons	30 MeV	3×10^{11}
Neutron stars	neutrons	300 MeV	3×10^{12}

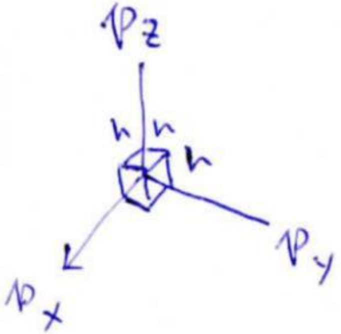
$$\boxed{\varepsilon_F = \frac{\hbar^2}{2m_e}\left(3\pi^2\, n_e\right)^{2/3}}$$

Considering the problem in terms of **momentum** , i.e., in the momentum space.

Degenerate State

$$E_n = \frac{\hbar^2}{2m}\left(\frac{n\pi}{\ell}\right)^2 \Rightarrow E_f = \frac{\hbar^2}{2m}\left(\frac{n_F\pi}{\ell}\right)^2 = \frac{\hbar^2}{2m}\left(3\pi^2 n_e\right)^{2/3}$$

Total $N_e = 2\cdot\frac{1}{8}\cdot\frac{4}{3}\pi n_F^3 = \frac{\pi}{3} n_F^3 \Rightarrow n_F = \left(\frac{3}{\pi} n_e\right)^{1/3}$



Uncertainty Principle $\Delta V \Delta^3 p \lesssim h^3$

$$2\cdot 4\pi p^2 dp = h^3 \cdot n_e(p)\, dp$$

Upto p_F, $2\cdot\frac{4}{3}\pi p_F^3 = N_e = n_e \cdot h^3 \Rightarrow p_F = \left(\frac{3h^3}{8\pi} n_e\right)^{1/3}$

Pressure and Momentum

$$\boldsymbol{P} = \frac{1}{3}\int_0^\infty v\, p\, n(p) dp$$

Non-relativistic

Pressure Integral $\mathbb{P} = \frac{1}{3}\int_0^\infty n(p)\, v\, p\, dp$ (use $v = p/m_e$)

$$= \frac{1}{3}\int_0^{p_F} \frac{8\pi p^2}{h^3}\,\frac{p}{m_e}\, p\, dp$$

$$= \frac{8\pi}{3 m_e h^3}\,\frac{1}{5}\, p_F^5 = \frac{8\pi}{15 m_e h^3}\, p_F^5$$

For electrons, $n_e = \frac{\rho}{\mu_e m_H}$ $\quad \therefore \mathbb{P} = \frac{h^2}{20 m_e}\left(\frac{3}{\pi}\right)^{2/3}\left(\frac{\rho}{\mu_e m_H}\right)^{5/3}$

In the non-relativistic case

$$\begin{aligned} P_{\mathrm{e,deg}}^{\mathrm{NR}} &= \frac{h^2}{20 m_e} \left(\frac{3}{\pi}\right)^{2/3} \frac{1}{m_{\mathrm{H}}^{5/3}} \left(\frac{\rho}{\mu_e}\right)^{5/3} \\ &= 1.00 \times 10^{13} \left(\frac{\rho}{\mu_e}\right)^{5/3} \quad [\mathrm{cgs}] \\ &\boxed{\propto \quad \rho^{5/3}} \end{aligned}$$

In the extremely relativistic case $v \rightarrow c$ in the pressure integral

$$\begin{aligned} P_{\mathrm{e,deg}}^{\mathrm{ER}} &= \frac{hc}{8} \left(\frac{3}{\pi}\right)^{1/3} \frac{1}{m_{\mathrm{H}}^{3/4}} \left(\frac{\rho}{\mu_e}\right)^{4/3} \\ &= 1.24 \times 10^{15} \left(\frac{\rho}{\mu_e}\right)^{4/3} \quad [\mathrm{cgs}] \\ &\boxed{\propto \quad \rho^{4/3}} \end{aligned}$$

For a composition devoid of hydrogen, and not very rich in extremely heavy elements, $\mu_e \approx 2$.

Gas Equation of State $P = P(\rho, T)$

In general, the pressure integral (momentum transfer)

$$\boxed{P = \frac{1}{3}\int_0^\infty v\,p\,n(p)\,dp}$$

For an deal gas $P \propto \rho T$

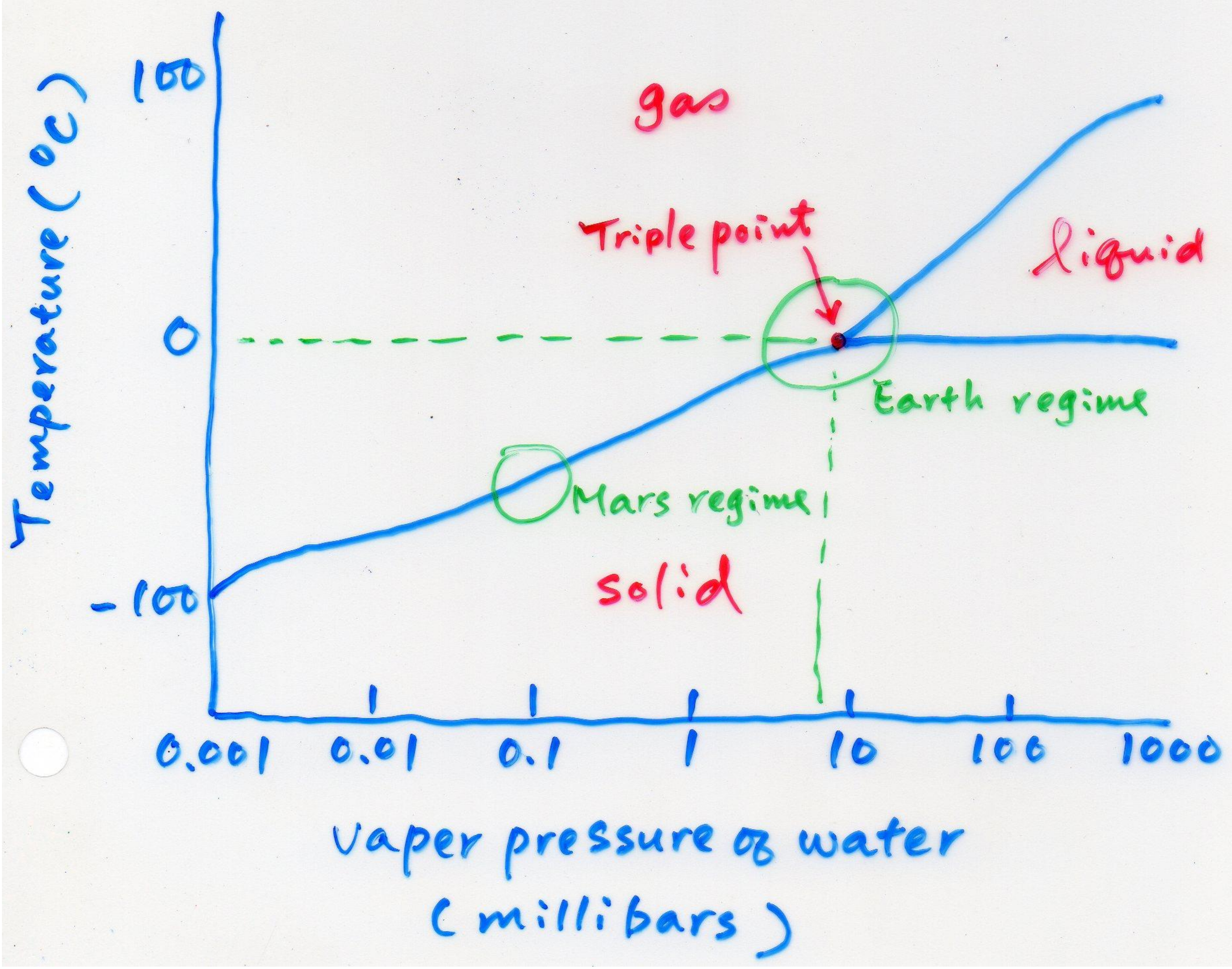
For a degenerate electron gas, P independent of T,

$P \propto \rho^{5/3}$ (non-relativistic)

$P \propto \rho^{4/3}$ (extremely relativistic)

Phase Diagram of Water

$\rho - T$ diagram



$$P_{\text{ideal gas}} \propto \rho\, T/\mu$$

$$P_{e,deg}^{NR} = 1.00 \times 10^{13} \left(\frac{\rho}{\mu_e}\right)^{5/3} \text{ [cgs]}$$

$$P_{e,deg}^{ER} = 1.24 \times 10^{15} \left(\frac{\rho}{\mu_e}\right)^{4/3} \text{ [cgs]}$$

$$P_{\text{rad}} = \frac{1}{3}\, a\, T^4$$

Non-Relativistic, Non-Degenerate (i.e., ideal gas)

Non-Relativistic, Extremely Degenerate

Extremely Relativistic, Extremely Degenerate

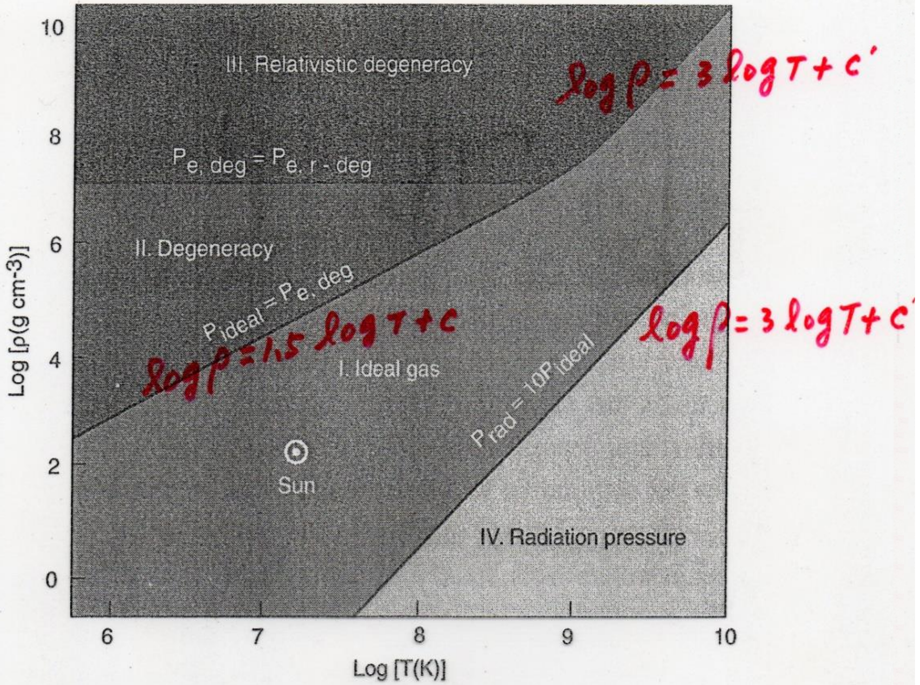


Figure 7.1 Mapping of the temperature-density diagram according to the equation of state.

NR, ND $\quad P \sim \rho T$

NR, ED $\quad P \sim \rho^{5/3}$

$\Big\}$ $\log \rho = 1.5 \log T + \text{const.}$

ER, ED $\quad P \sim \rho^{4/3}$ $\quad (\sim \rho T)$ $\Big\}$ $\log \rho = 3 \log T + \text{const}$

Prad vs Pideal gas $\quad P_{rad} \sim T^4$, $\quad P_{gas} \sim \rho T$ $\Big\}$ $\log \rho = 3 \log T + \text{const}$

Mass-radius relation for a degenerate electron gas $P \sim \frac{M^2}{R^4}$

In the NR case, $P \propto \rho^{5/3} \sim \left(\frac{M}{R^3}\right)^{5/3} = \frac{M^{5/3}}{R^5} \Longrightarrow \boxed{MR^3 = \mathrm{const}}$

So $M \nearrow$, $R \searrow$, $\rho \nearrow\nearrow$, electrons move ever faster.

$$\log\left(\frac{R}{R_\odot}\right) = -\frac{1}{3}\log\left(\frac{M}{\mathrm{M}_\odot}\right) - \frac{5}{3}\log\mu_e - 1.397$$

In the ER case, $P \propto \rho^{4/3} = \frac{M^{4/3}}{R^4}$, $\underline{\text{no}}$ solution between M and R.

A mass limit for a degenerate electron body (white dwarf)

Chandrasekhar limit $M_{WD} \lesssim 5.8\, M_\odot/\mu_e^2$

$$L = \sigma T_e^4 \left(4\pi R^2 \right)$$

$$\log\left(\frac{L}{L_\odot}\right) = 4 \log\left(\frac{T_e}{T_{e\odot}}\right) + 2 \log\left(\frac{R}{R_\odot}\right)$$

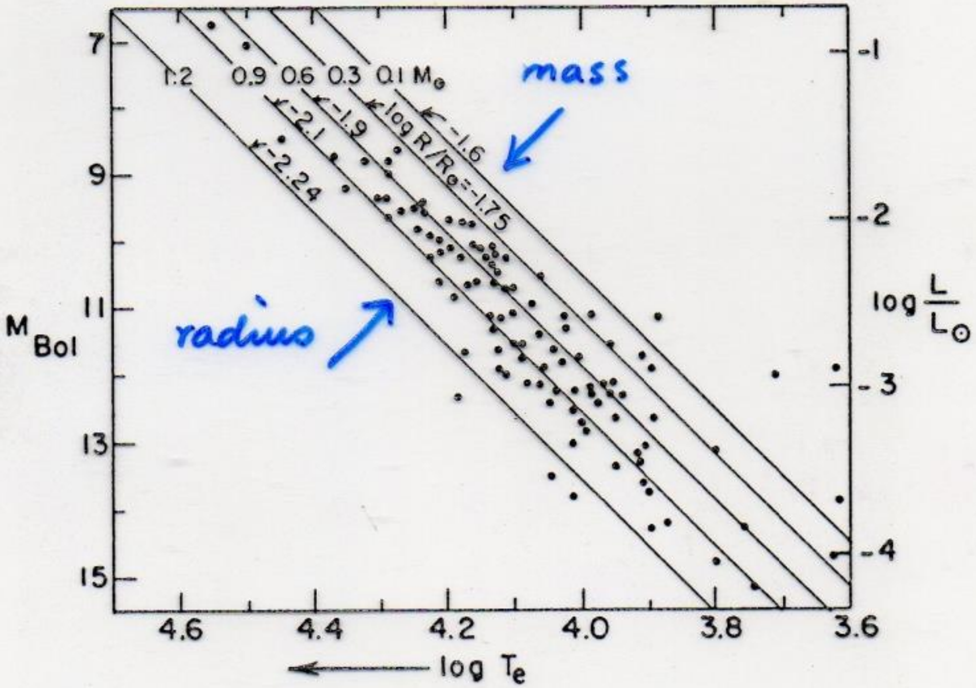
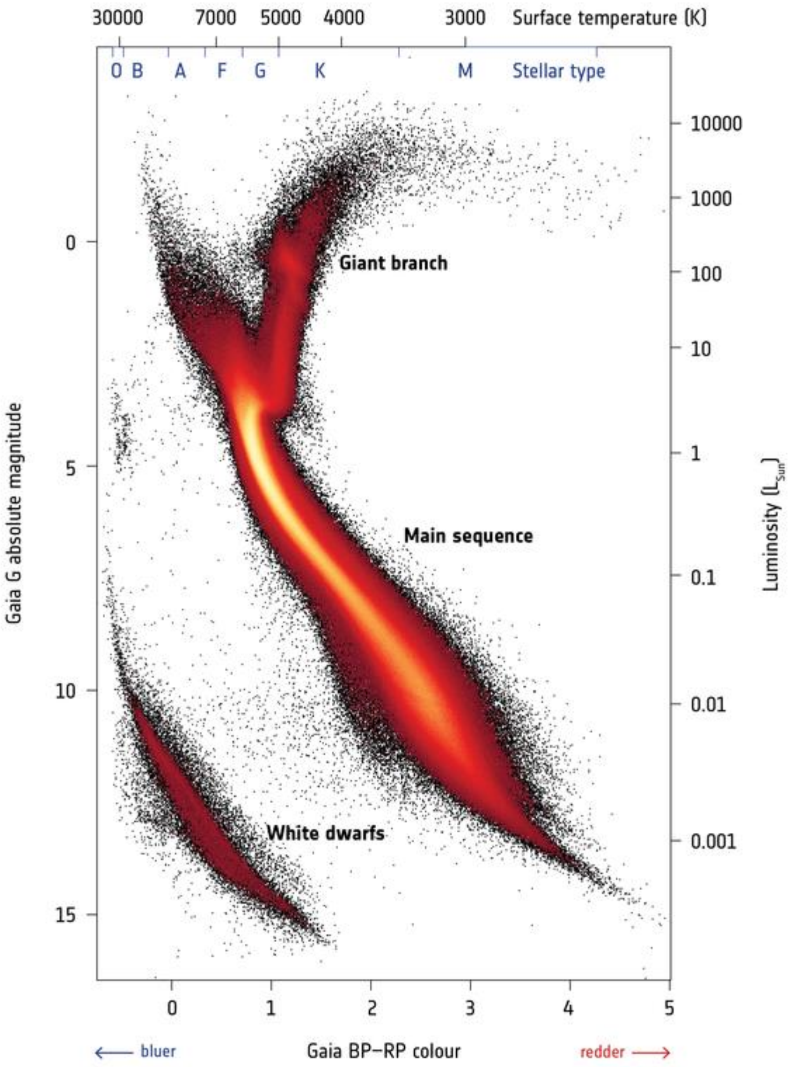
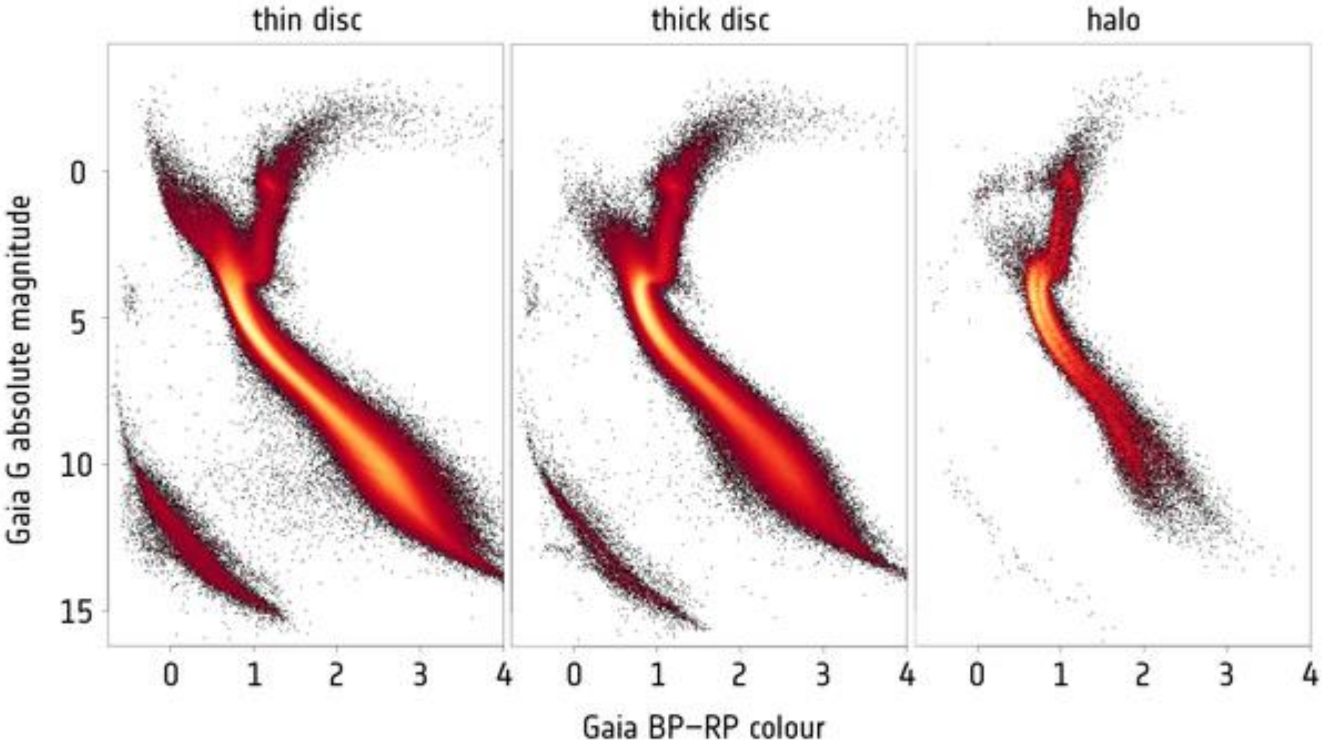
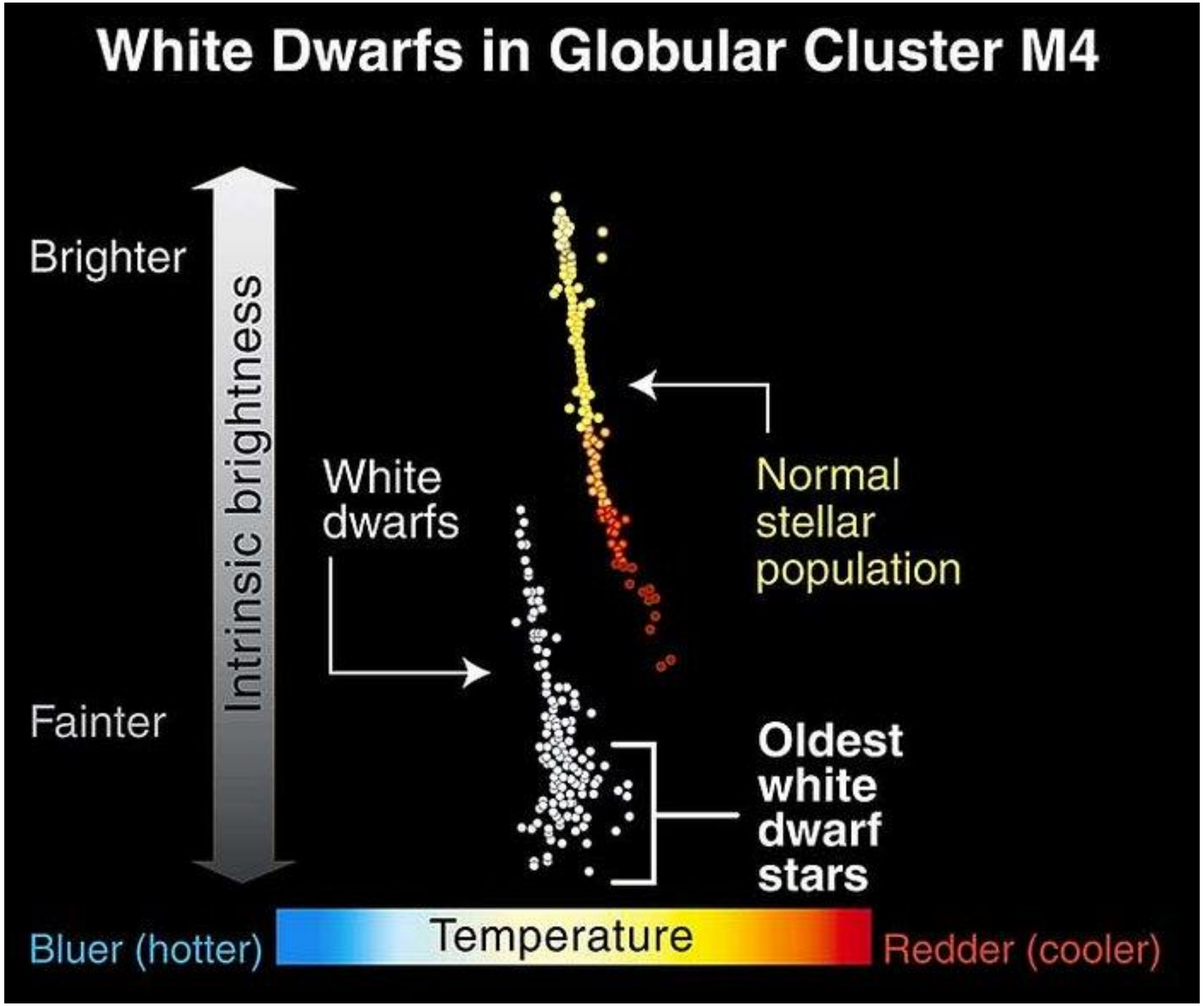


FIGURE 8.14. White dwarf Hertzsprung–Russell diagram. Lines of constant radius are shown. Also shown are the masses based on completely degenerate core models containing elements having $\mu_e = 2$ (after Weidemann (We68)). Reprinted with permission from *Annual Review of Astronomy and Astrophysics*, Vol. 6, ©1968 by Annual Reviews, Inc.).

→ GAIA'S HERTZSPRUNG-RUSSELL DIAGRAM







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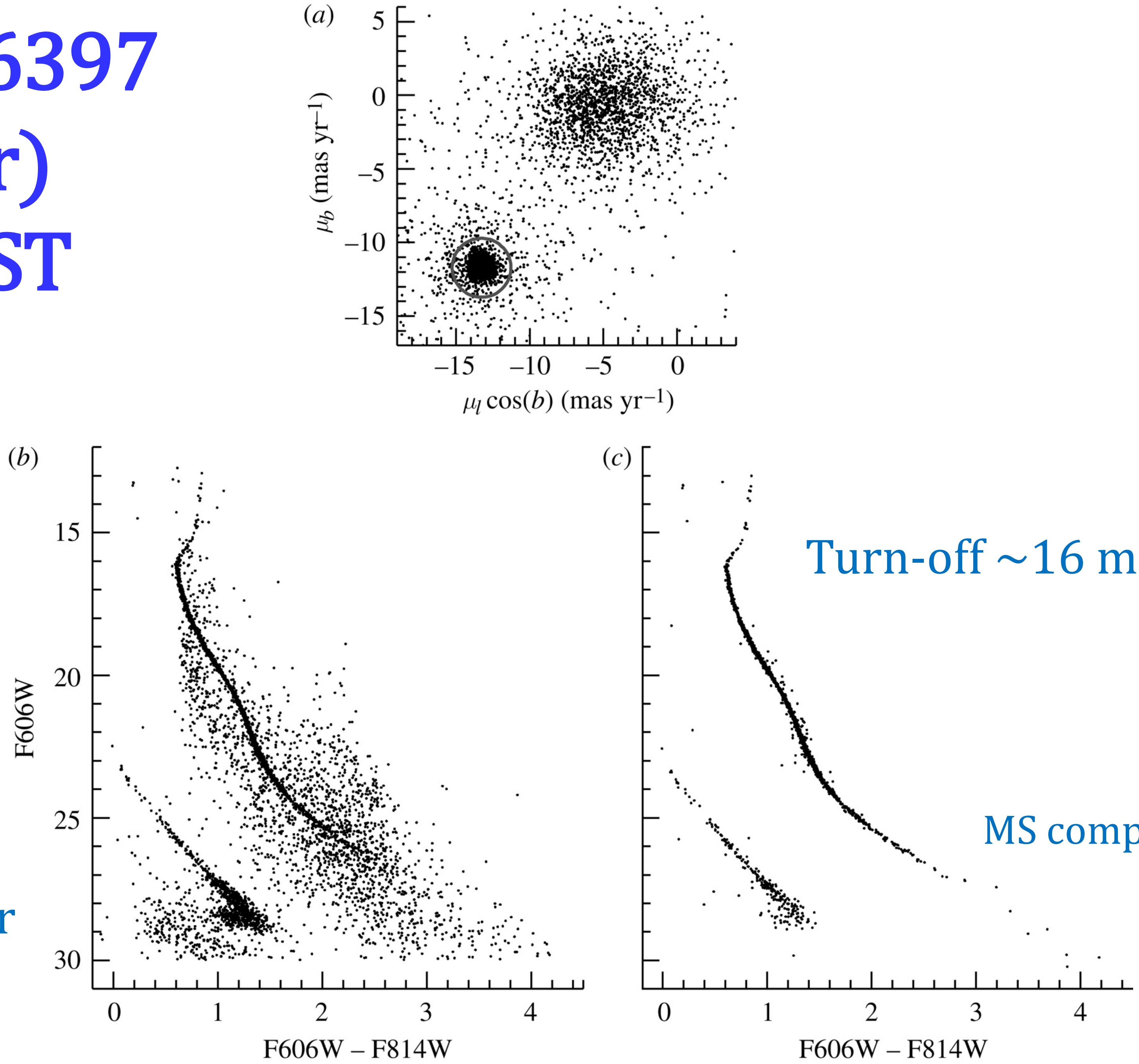
GC NGC 6397 (~12 Gyr) by the HST

Kalirai 2010



Turn-off ~16 mag, 0.8 $M_\odot$

MS complete ~26 mag, 0.092 $M_\odot$

Detection limit ~30 mag, 0.083 $M_\odot$

WD cooling:
11.47 ± 0.47 Gyr