估計在中美掩星計畫中
古柏帶星體遮掩到背景恆星的機率

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摘要

中美掩星計畫的目的是要用偵測遮掩背景恆星的機會來估計古柏帶星體(KBOs)的數量。然而偵測到掩星事件的機率和實際情況與偵測能力有關。從實際情況來看，最主要的參數包含了古柏帶星體的分布(我們想知道的)、背景恆星的數量密度及角直徑大小的分布、古柏帶星體與地球的相對速度造成在像場裡面的陰影速度。而偵測事件發生的能力跟觀測的儀器和當時環境有關，例如 CCD 的積分時間、取樣速率、月象、氣候的變化、繞射效應等..以及需要更多方面的模擬來探討這些參數。這邊我們報告在幾何方法給定時間及一些參數下去計算掩星事件機率大小。

Probability of Stellar Occultation by KBOs
in the TAOS Project

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Abstract

The TAOS project aims to estimate the number of Kuiper-belt objects (KBOs) by detecting chance stellar occultation. The detected rate of such occultation events depends on the actual occurrence rate and the detectability. For the former, the relevant parameters include the distributed population of the KBOs (our goal), the surface number density and angular size distribution of background stars, and the relative shadow speed. The detectability of any event depends on the instruments and circumstances, such as the CCD integration time, sampling rate, moon phase, sky variations, diffraction effect, etc., and requires an extensive simulation to explore the parameter space. Here we report the computation of the probability of KBO occultation from a geometric consideration. With a density of 1000 stars and 10^4 to 10^6 KBOs deg^{-2}, a relative velocity about 0".001 s^{-1}, one would expect about 10^{-8} events per hour of observations.

關鍵字（Keywords）：古柏帶星體（Kuiper Belt objects）、掩星（occultation）

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1. Introduction

The TAOS (Taiwanese-American Occultation Survey) project (King et al. 2001; Chen et al. 2003; Alcock et al. 2003; Lehner et al. 2006) aims to determine the number of small Kuiper-belt objects (KBOs) by detecting chance stellar occultation events.

Most of these events last < 1 second. This project makes use of four fast (f/1.9) optics 0.5m telescopes, each equipped with a 2048×2048 pixel CCD camera rendering 3 deg^2 FOV, located at the Lulin Observatory. After one year of operation, some 10^9 photometric measurements have been collected. No event has been detected so far, implying a genuine deficit of kilometer sized KBOs compared to an extrapolation from the larger (> 100 km) sized population (Chen et al., 2006). Our eventual goal is to estimate the KBO population from the observed event rate. Relevant parameters include (1) the size distribution of the KBOs (our goal), (2) the surface number density and angular size distribution of backgrounds, (3) the relative shadow speed, e.g., observing in the opposition versus in quadrature, and (4) the detectability of an event limited by the instruments (e.g., the limiting magnitude, data sampling rate, etc.) and circumstances (e.g., moon phase, sky variations, diffraction effect, etc.) The overall efficiency of our observations can be computed only with extensive simulation. Here we present a simple derivation of the expected occultation event rate by geometrical consideration.

2. Computation of Parameters

We start the calculation of the probability of stellar occultation by KOBs by first estimating the chance of at least one background star being covered by the trajectory of one KBO. In Fig. 1, within the field of view (FOV) of h^2, there is one KBO with an angular diameter of θ_k^j and one background star with an angular diameter of θ_*^i. When the positions of the KBO and the star are randomly distributed within the FOV, the probability of an occultation, i.e., the KBO shall cover any part of the star, depends on the sky areas of the KBO and of the star. The occultation probability therefore is simply the fraction within the FOV of the shaded area, which with a diameter of $\theta_*^i + \theta_k^j$ represents the region within which the two bodies overlap. Designating the occultation event set as O_{ij}, we consider the complement occultation event set O_{ij}^c for which the KBO j and the star i do not overlap each other at all. In such a non-occultation case, the probability is,

$$P'(\mathrm{O}_{ij}^c) = 1 - P'(\mathrm{O}_{ij}) = 1 - \frac{\pi\left(\theta_k^j/2 + \theta_*^i/2\right)^2}{h^2} \quad (1)$$

In reality, the KBO should move across the FOV, so we now consider the KBO with a trajectory, as depicted in Fig. 1. This simply enlarges the effective area of the KBO by adding the rectangular region swept by the KBO with an angular speed V during time t. The non-occultation probability now becomes

$$P(\mathrm{O}_{ij}^c) = 1 - P(\mathrm{O}_{ij}) = 1 - \left(\frac{\pi\left(\theta_k^j/2 + \theta_*^i/2\right)^2}{h^2} + \frac{\left(\theta_k^j + \theta_*^i\right)Vt}{h^2}\right) \quad (2)$$

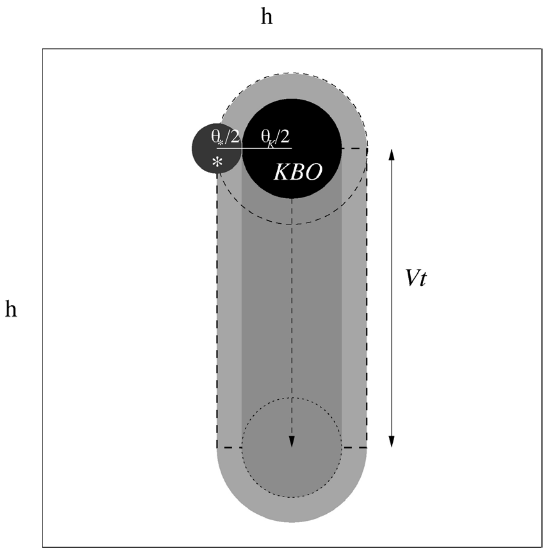

Fig. 1: (a)A stellar occultation occurs when the KBO (with a diameter of θ_k) covers at least part of the background star (with a diameter of θ_*), or equivalently as the center of the star falls within the shaded "action" region. (b)As the KBO moves relative to the star with a speed V, the effective area for occultation in a duration of time t becomes larger by an additional rectangular region swept by the "action" region.

We now consider the case of more than one star in the FOV. With N_* background stars, each with an angular size θ_*^i , the non-occultation event set O_j^c -- namely none of the stars is occulted by the KBO -- is the intersection of all O_{ij}^c s. The corresponding probability is $P(\mathrm{O}_j^c) = P(\mathrm{O}_{1j}^c \cap \mathrm{O}_{2j}^c \cap ... \cap \mathrm{O}_{N_*j}^c)$ which, because every occultation event is independent, equals to the product of all the probabilities, i.e., $P(\mathrm{O}_{1j}^c) \cdot P(\mathrm{O}_{2j}^c) \cdot ... \cdot P(\mathrm{O}_{N_*j}^c)$. Therefore, for one KBO and N_* background stars, the non-occultation probability, i.e., the chance that the KBO does not occult any of the stars in the FOV, becomes

$$P(\mathrm{O}_j^c) = \prod_{i=1}^{N_*} P(\mathrm{O}_{ij}^c)$$
$$= \prod_{i=1}^{N_*} \left\{ 1 - \left(\frac{\pi\left(\theta_k^j/2 + \theta_*^i/2\right)^2}{h^2} + \frac{\left(\theta_k^j + \theta_*^i\right)Vt}{h^2} \right) \right\} \quad (3)$$

That is, the probability of a particular KBO j in the FOV to occult at least one background star is,

$$P(\mathrm{O}_j) = 1 - P(\mathrm{O}_j^c) \quad (4)$$

where O_j is the occultation event for a particular KBO j.

Now we consider two KBOs in the FOV. The occultation event set O in this case is the union of the two independent sets O_1 and O_2 , so the probability $P(\mathrm{O})$ is $P(\mathrm{O}_1 \cup \mathrm{O}_2)$. When the case is generalized to N_k KBOs, instead of the probability of the union, we compute the probability of the intersection. In the two-KBO case, by use of the De Morgan's law,

$$P(\mathrm{O}_1 \cup \mathrm{O}_2) = 1 - P\left[(\mathrm{O}_1 \cup \mathrm{O}_2)^c\right]$$
$$= 1 - P(\mathrm{O}_1^c - \mathrm{O}_2^c)$$
$$= 1 - P(\mathrm{O}_1^c)P(\mathrm{O}_2^c)$$

Similarly, if there are N_k KBOs, the overall probability can be written finally as

$$P(\mathrm{O}) = 1 - \prod_{j=1}^{N_k} P(\mathrm{O}_j^c)$$
$$= 1 - \prod_{j=1}^{N_k}\prod_{i=1}^{N_*} \left\{ 1 - \left(\frac{\pi\left(\theta_k^j/2 + \theta_*^i/2\right)^2}{h^2} + \frac{\left(\theta_k^j + \theta_*^i\right)Vt}{h^2} \right) \right\} \quad (5)$$

Eq. 5 is what we aimed to calculate, namely the probability of at least one occultation event for N_k KBOs and N_* stars in the field of view. In the following we discuss each relevant parameter that goes into the computation of Eq. 5.

2.1 Angular Size of the KBO, θ_k

The angular size of a KBO depends on its physical size and the geocentric distance. The latter varies little given the large distances of the KBOs considered here (~ 40 AU). Fig. 2 shows the configuration of the KBO, Earth, and the Sun. For a KBO with a heliocentric distance of D AU, its geocentric distance X (in AU) at a position angle α is $X=\sqrt{1+D^2-2D\sin\alpha}$. The position angle α, the ecliptic longitude difference between the Earth and the KBO, is defined such that α =0 at opposition and α =90 deg at quadratures. Accordingly, the angular diameter in milli-arcsecond (mas) of a KBO with a physical radius of R_k (in km) is,

$$\theta_k = 2.7575(\frac{R_k}{\text{km}})(\frac{\text{AU}}{X}) \quad [\text{mas}] \tag{6}$$

For a KBO with R_k = 1 km at D = 42 AU, its angular size $\theta_k \sim 0.067$ mas.

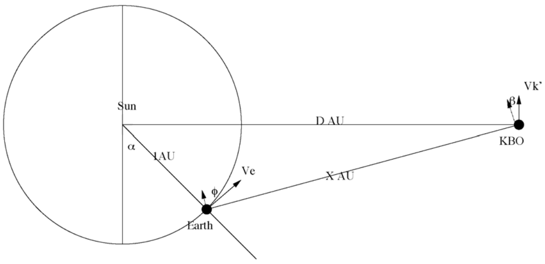

Fig. 2: The relative positions of the Sun, Earth and a KBO located at *D* AU from the Sun. The geocentric distance of the KBO is denoted as *X*. The solid arrows denote the tangential velocities of the Earth and the KBO, whereas the dash arrows show the direction perpendicular to our line of sight to the KBO.

2.2 Angular Size of Background Stars, θ_*

In stellar occultation by KBOs, the angular size of a background star is a critical parameter. The stellar size affects the occultation probability, the duration and fractional flux drop of an event, and whether the diffraction effect is important. The event duration is related to $(\theta_k+\theta_*)/V_k$, whereas the flux drop is proportional to $(\theta_*/\theta_k)^2$. A stellar size of 1 mas corresponds to a projected linear size of ~ 30 km at 42 AU. Occultation of such a star by a KBO of a few km therefore does not produce a flux drop detectable by the TAOS system. On the other hand, when the KBO angular size is comparable to the stellar angular size, the diffraction effect should be considered. A reliable estimation of the angular size of every observed star in a TAOS is hence necessary.

The Barnes-Evans Surface Brightness (SB) technique provides a way to effectively estimate angular diameter of a star from photometric measurements alone (Barnes & Evans 1976). The SB relations link the emerging flux per solid angle from a star to its color, or effective temperature. The SB (F_λ) is related to the unreddened apparent magnitude ($m_{\lambda 0}$) and the angular diameter, θ_*,

$$F_\lambda = 4.2207 - 0.1 m_{\lambda 0} - 0.5\log\theta_* \tag{7}$$

which is applicable not only in visual wavelengths but also in other wavelengths (van Belle, 1999; Nordgren et al., 2002; Kervella et al., 2004; Di Benedetto, 2005). The slope and zero-point of the linear SB-color relation could be well calibrated direct stellar angular-diameter measurements with long-baseline interferometry or lunar occultation observations (van Belle 1999; Nordgren et al. 2002; Kervella et al. 2004; Di Benedetto 2005). For example, with a sample of 57 giants observed with the Naval Prototype Optical Interferometer (NPOI), Nordgren et al. (2002) established the empirical relation between the SB and the $(J-K)_0$ color index,

$$F_{K_0} = (3.942\pm0.006)-(0.095\pm0.007)(J-K)_0 \tag{8}$$

From the new high-precision angular-diameter measurements with the VLT interferometer, Kervella et al. (2004) established the empirical relations between the limb darkened angular diameters and visual-infrared color indices, and found that the SB relations for giants, dwarfs and subgiants are almost similar. This means that two stars of different luminosity classes but with similar magnitudes in two bands will appear to have approximately the same angular diameters. In our computation of the stellar angular diameters for the TAOS fields, we thus made no distinction of the luminosity class of a star.

The near-IR SB technique shows relatively less intrinsic dispersion compared to that in visual wavelengths. Moreover, the relation is less affected by interstellar extinction in derivation of the unreddened colors. Given the empirical reddening law, $A_J = 0.28A_V$ and $A_K = 0.11A_V$ (Rieke & Lebofsky 1985), the angular diameter estimation of unknown reddened star from the near-IR relations is less affected, particularly for nearby stars.

The 2MASS point source catalog provides us a uniform dataset in the near-IR J, H, K_s bands with high photometry accuracy (Cutri et al. 2003). We have used it to estimate the angular diameters of field stars in the TAOS images by using Eq. 7 and Eq. 8. Stars in a TAOS image are cross-matched with the 2MASS catalog with a positional coincidence of less than 2". Fig. 3 shows the angular-diameter distribution for the two TAOS fields (#120 and #121). Field #120 has a total of about 1330 stars brighter than R ~ 16 mag in the TAOS 3 deg^2 FOV. Field #121 is near the Galactic plane and is much more crowded, with a total of about 2700 stars for the same magnitude limit.

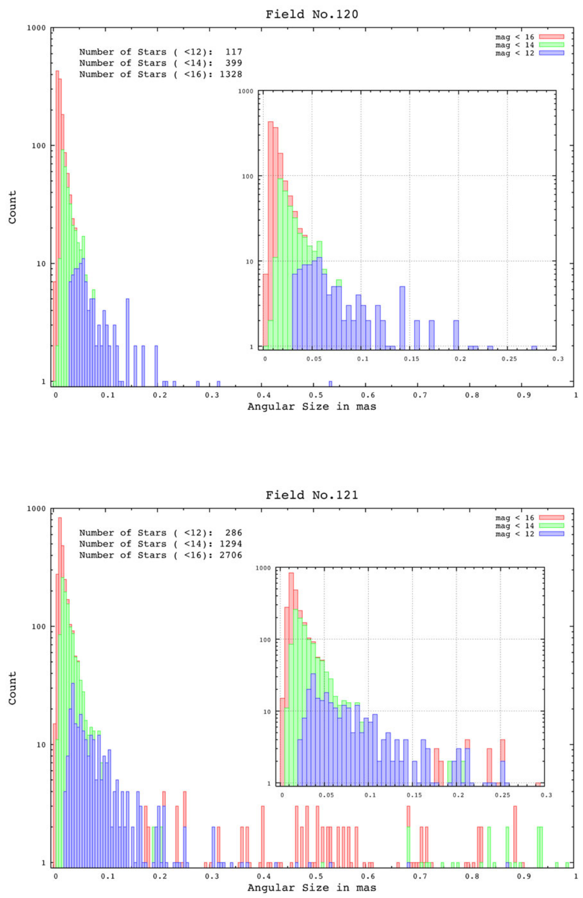

Fig. 3: Distribution of angular diameters of stars in TAOS fields #120 (above) and #121 (below).

2.3 Relative Speed of a KBO, V_k

A KBO has an orbital speed of $\sim 30/\sqrt{D}$ km s^{-1}. Near the opposition, the sky motion of the KBO is dominated by the earth's own orbital motion. In this case, V_k is maximal, and so is the chance of stellar occultation by the KBO. As a reference, a KBO with a retrograde speed of 4 km^{-1}, would ~ 34 km s^{-1} relative to Earth, and such a KBO of 2 km across would blink out a background star for 0.06 s. On the other hand, the sky motion of the KBO is nearly minimal near the quadratures; the chance of stellar

Table 1: Stellar Occultation Probability per Hour for 1-km KBO

Distance [AU]	Radius [km]	Position Angle [deg]	Field No. (# of stars)	Probability 1 KBO
42	1	90	#120 (399)	4.5584×10^{-9}
			#121 (1511)	2.0728×10^{-8}
		0	#120 (399)	1.2078×10^{-10}
			#121 (1511)	5.5552×10^{-10}

occultation hence is low but any occultation event would last for a longer time, therefore easier to detect, than in the opposition case.

In general, the relative angular speed V_k of a KBO is

$$V_k = 41.0956 \frac{D \sin\alpha - 1 - D^{-1/2}\cos\alpha}{X^2} \quad [\text{mas s}^{-1}] \quad (9)$$

The detailed derivation of Eq. 9 is given in the Appendix.

3. Expected Event Rate by TAOS

We are now ready to compute the expected event rate on the basis of the TAOS parameters estimated in the last section. These parameters are:

- **Field of View** (*h*): For the TAOS telescopes, h = 1.7deg = 1.7 × 3.6 × 10^6 mas.
- **Observation Time** (*t*): We adopt a basic unit of *t*=1 hr. To estimate the expected number of events per night, one then simply scales to, for example, an 8-hour night, or by taking into account the duty cycle of the observing sessions, to estimate the expected number of events in a year.
- **Position Angle of the KBO** (α): This affects the relative velocity V_k. We consider the cases at opposition and at quadrature.
- **Number of Background Stars** (N_*): Here we consider two star fields, #120 and #121 as an example for a sparse field and a crowded field, respectively. The TAOS limiting magnitude is R ~ 14 mag, set empirically by the photometric analysis pipelines, so $N_* \cong 400$ for Field #120, and $N_* \cong 1300$ for Field #121 (cf. Fig. 3).
- **KBO Parameters**: We consider the case for R_k = 1 km, located at a heliocentric distance of *D*=42 AU.

The results of our calculation are given in Table 1. The first and second columns are the heliocentric distance (*D*) and the radius (R_k) of the KBO under consideration. The third column is the phase angle of the earth and the KBO relative to the Sun (α). The fourth column is the TAOS field No. The fifth column is our estimated probability of stellar occultation for *one* KBO in the particular TAOS field per hour. For each set of parameters, the maximum event rate is marked in bold. As expected, the largest probability always occurs for Field #121, which has a denser stellar field, and for opposition, which has a faster relative speed between the KBO and background stars.

It has been suggested that medium-sized KBOs (tens to hundreds of km) are depleted relative to the large (> 100 km) ones, perhaps as a consequence of collisional destruction (Bernstein et al. 2004) [see Fig. 4]. To determine whether such a deficit extends to the km-sized KBOs therefore is important and is the primary goal of the TAOS project. For the classical KBO population, the extrapolated surface number density ranges from $10^{3.5}$ to $10^{5.5}$ deg^{-2} (Bernstein et al. 2004) [cf. Fig. 4]. With the maximum of a few times 10^{-8} events per hour, as shown in Table 1, one would expect 0.1 to 10 events in a year, assuming a 100-night year and 8-hour nights. The probability can be much smaller with unfavorable parameters (e.g., away from opposition or sparse star fields).

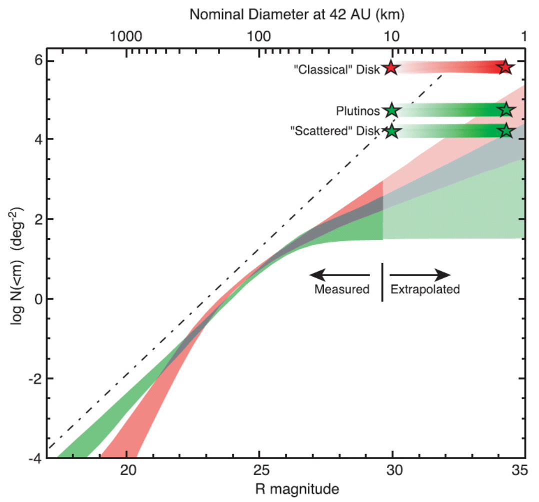

Fig. 4: Cumulative surface density of the excited (green) and the classical KBO (red) populations (figure taken from Bernstein et al. 2004).

So far TAOS has collected some billion photometric measurements. No events were detected with high significance. As noted earlier, when the angular size of the KBO becomes small (either physically small or more distant) so as to be comparable to that of the star, diffraction effect starts to kick in, and our geometric assumption is no longer valid. In addition, a partial occultation has a reduced fractional flux drop and a shorter event duration, both of which are unfavorable for the detectability of an event. A recent X-ray result suggests that 100-m sized KBOs are far from depletion (Chang et al. 2006), perhaps as high as ~ 2×10^{12} deg^{-2} (Chang 2006, private communication). Extrapolation from the smaller size end, the number density for 1 - 10 km KBOs could be as high as 10^{9} deg^{-2}. Such an event rate was not seen in either the X-ray data or the TAOS data collected so far. The TAOS project obviously needs to continue collecting data and requires a comprehensive detectability analysis to resolve the inconsistency.

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Appendix

In Fig. 2, the Earth orbit velocity is,

$$V_e = \frac{2\pi D_{\oplus}}{T_{\oplus}} = \frac{2\pi 1}{1} = 2\pi\left[\mathrm{AU}\cdot\mathrm{yr}^{-1}\right] \quad \text{(A1)}$$

According to the Kepler's third law, we compute the period of the KBO locates at D AU is $D^{3/2}$. Use this result, we find the velocity of KBO is,

$$V_{k'} = \frac{2\pi D_k}{T_k} = \frac{2\pi D_k}{D_k^{3/2}} = 2\pi D^{-1/2}\left[\mathrm{AU}\cdot\mathrm{yr}^{-1}\right] \quad \text{(A2)}$$

Use this two equations, we estimate the difference of the velocity components which are dash arrows in the Fig. 2. Then the relative velocity is,

$$\begin{aligned} V_k &= V_e\cos\phi - V_{k'}\sin\beta \\ &= 2\pi\left[1\cdot\cos\phi + D_k^{-1/2}\cdot\cos(\phi+\alpha)\right]\left[\mathrm{AU}\cdot\mathrm{yr}^{-1}\right] \end{aligned} \quad \text{(A3)}$$

where $\beta = \phi - (\frac{\pi}{2} - \alpha)$.

It is convenient to use the unit in milli-arcsecond per second. Use the equation (A3),

$$\begin{aligned} V_k &= \frac{2\pi\cdot 3.0857\times 10^{13}\left[\cos\phi + D_k^{-1/2}\cos(\phi+\alpha)\right]}{1.4960\times 10^{8}\cdot\sqrt{1+D_k^2-2D_k\sin\alpha}}\left[\mathrm{arcsec}\cdot\mathrm{yr}^{-1}\right] \\ &= \frac{2\pi\cdot 3.0857\times 10^{13}\left[\cos\phi + D_k^{-1/2}\cos(\phi+\alpha)\right]\times 10^{3}}{3\times 10^{7}\times 1.4960\times 10^{8}\cdot\sqrt{1+D_k^2-2D_k\sin\alpha}}\left[\mathrm{mas\,s}^{-1}\right] \\ &= \frac{41.0956\left[\cos\phi + D_k^{-1/2}\cos(\phi+\alpha)\right]}{\sqrt{1+D_k^2-2D_k\sin\alpha}}\left[\mathrm{mas\,s}^{-1}\right] \end{aligned} \quad \text{(A4)}$$

Also, we find out the relation of ϕ and α, in the triangle made up by the Sun, the Earth and the KBO, $D_k\cdot\sin(90-\alpha) = X\sin\phi$ is the height of the triangle use the Sun and the Earth connection as the base; $X\cdot\sin\phi$ is the same one. Since they are equal, we can get the relation of them below,

$$D_k\sin(\frac{\pi}{2}-\alpha) = X\sin\phi$$

$$\sin\phi = \frac{D_k\cos\alpha}{\sqrt{1+D_k^2-2D_k\sin\alpha}} \quad \text{(A5)}$$

$$\cos\phi = \frac{D_k\sin\alpha - 1}{\sqrt{1+D_k^2-2D_k\sin\alpha}} \quad \text{(A6)}$$

$$\begin{aligned} V_k &= 41.0956\frac{D_k\sin\alpha - 1 - D_k^{-1/2}\cos\alpha}{1+D_k^2-2D_k\sin\alpha}\left[\mathrm{mas\,s}^{-1}\right] \\ &= 41.0956\frac{D_k\sin\alpha - 1 - D_k^{-1/2}\cos\alpha}{X^2}\left[\mathrm{mas\,s}^{-1}\right] \end{aligned} \quad \text{(A7)}$$